SIGN-BIT AMPLITUDE RECOVERY IN GAUSSIAN NOISE

L.M. HOUSTON, G.A. GLASS and A.D. DYMNIKOV

The University of Louisiana at Lafayette, Lafayette, LA 70504-4210, U.S.A.
houston@louisiana.edu

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ABSTRACT

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Sign-bit amplitude recovery implies the recovery of signal from the average of the sign-bits of signal plus noise. We show that, given a Gaussian noise density, the average of the sign-bits of signal plus noise is not the signal, but is the Gauss error function with an argument that is proportional to the signal and inversely proportional to the standard deviation of the noise. This result can appear to provide amplitude recovery by producing a facsimile of the signal but the signal is only properly recovered by processing the data with the inverse error function. Based on the Central Limit Theorem, the optimal signal-to-noise ratio for amplitude recovery in Gaussian noise is identical to that of uniform noise, S/N = 1. This theory is tested using computer simulations with synthetic signal and noise. First, we demonstrate sign-bit amplitude recovery in uniform noise. Next, we compare the sign-bit average in uniform noise with the sign-bit average in Gaussian noise before and after the inverse error function is applied. Finally we compare hard clipping in uniform noise to soft clipping in Gaussian noise which occurs for large signal-to-noise ratios.

KEYWORDS: sign-bit data, error function, probability density, Gaussian, variance.

INTRODUCTION

Seismic data acquisition generally consists of noisy data which are collected in a redundant manner. Coherent signals are subsequently recovered by summing over the data to reduce the noise amplitude and increase the signal amplitude. Interestingly, for relatively large noise amplitudes, only the signs of the data are needed to recover coherent signal. In contrast, the signal is clipped

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if the noise amplitude is too low. We point out that signal recovery does not require true amplitude. Sign-bit recovered signal is generally modulated by a scaling factor related to the noise.

Historically, the amplitude recovery of sign-bit data has been exploited to reduce the required dynamic range of seismic recordings. Sign-bit digital recording means that only the sign of the true amplitude sample is recorded with one bit. In conventional seismic recording, 16 to 20 binary bits per sample point (O'Brien et al., 1982) are recorded. The economic advantages of sign-bit acquisition are immediately obvious. Although it was more widely used during earlier years of the seismic industry, sign-bit recording remains as a viable tool today (de Ridder and Prieto, 2008). Nonetheless, sign-bit recording is a tool that is suitable only for a large number of channels.

Sign-bit amplitude recovery is well documented (O'Brien et al., 1982). However, as stated by O'Brien et al. (1982), "we feel that there is room for considerably more work, especially in extending the theory to more physically realistic noise distributions". This paper aims to extend the theory of sign-bit amplitude recovery by applying sign-bit amplitude recovery to Gaussian noise in direct comparison to sign-bit amplitude recovery in uniform noise. As reported by O'Brien et al. (1982), there is no experimental evidence which selects a particular noise distribution as being the most applicable. Because the results for a continuous treatment of uniform noise have been consistent (Houston and Richard, 2003) with the results reported by O'Brien et al. (1982) who considered discrete noise, we also treat this problem with continuous noise.

The description of the recovery of the amplitudes from sign-bit data by O'Brien et al. is using a simplified statistical model of sign-bit recording, i.e., the "flipping-of-a-coin-model", where the random background noise recorded by a geophone represents the "true coin", i.e., summing the random noise gives zero as the result, while the coherent signal represents the bias which (by "sufficient repetition" and an actual Signal-to-Noise Ratio $S/N \leq 1$, i.e., the signal amplitude is not allowed to exceed the noise amplitude) will be recovered to the required precision.

If the noise amplitude is exceeded by the signal amplitude, clipping will occur, i.e., the result will be an incomplete amplitude recovery of the signal. But clipping is not the only problem: note that in order to make (as O'Brien et al. state) "sign-bit digitization a completely viable technique for recording seismic data" the two conditions stated by O'Brien et al. (1982) , i.e., $S/N \leq 1$ as well as sufficient redundancy, are not enough. The third condition, which is required, is that of no residual timeshifts. Since this cannot be assured for CMP-stacking, the sign-bit technique is not well suited for application in the CMP-domain, because the resulting bias (= signal) will be distorted by these static shifts.

In this paper we use two different approaches to derive the expectation value of the sign of signal plus noise for both uniform noise and Gaussian noise. These expectations are then used to determine optimal sign-bit amplitude recovery. Finally, we test the theory with computer simulations of synthetic signal and noise.

SIGN-BIT AMPLITUDE RECOVERY

Sign-bit amplitude recovery occurs when the sum of the signs of signal plus noise is proportional to the signal. Sign-bit amplitude recovery of a signal f can be described in continuous noise as follows. [The following presentation is consistent with Houston and Richard (2003)]. The average or expectation of the sign-bits is

$$E[\mathrm{sgn}(f + X)] = \int_{-\infty}^{+\infty} \mathrm{sgn}(f + x)\rho(x)dx \quad , \tag{1}$$

where $\rho(x)$ is the probability density of the noise, modeled by the random variable X, and sgn is the signum function (Gabel and Roberts, 1987) given by

$$\left\{ \begin{array}{lll} \mathrm{sgn}(y) = +1, & y > 0 \\ \mathrm{sgn}(y) = 0, & y = 0 \\ \mathrm{sgn}(y) = -1, & y < 0 \end{array} \right\} \quad . \tag{2}$$

The integral in (1) divides into a positive region and a negative region:

$$E[\mathrm{sgn}(f + X)] = \int_{-f}^{+\infty} \rho(x)dx - \int_{-\infty}^{-f} \rho(x)dx \quad . \tag{3}$$

Let $\rho(x)$ be symmetric about $x = 0$. Then

$$\int_{-\infty}^{-f} \rho(x)dx = \int_{f}^{\infty} \rho(x)dx \quad . \tag{4}$$

So

$$E[\mathrm{sgn}(f + X)] = \int_{-f}^{\infty} \rho(x)dx = \int_{\infty}^{f} \rho(x)dx \quad , \tag{5}$$

which reduces to

$$E[\mathrm{sgn}(f + X)] = \int_{-f}^{f} \rho(x)dx \quad . \tag{6}$$

For example, consider the case of uniform noise. Let the probability density be that of a uniform noise distribution (Rice, 1995)

$$\rho(x) = \begin{cases} 1/2a \ , & -a \leq x \leq a \\ 0 \ , & \text{else} \end{cases} \quad . \tag{7}$$

For $|f| > a$, $\mathrm{sgn}(f + X) = \mathrm{sgn}(f)$ and the data is clipped.

For $|f| \leq a$, eq. (6) then becomes

$$E[\mathrm{sgn}(f + X)] = \int_{-f}^{f} (1/2a)dx = f/a \quad . \tag{8}$$

Eq. (8) is equivalent to sign-bit amplitude recovery.

The variance is computed as follows:

$$\mathrm{Var}[\mathrm{sgn}(f + X)] = E[\mathrm{sgn}^2(f + X)] - \{E[\mathrm{sgn}(f + X)]\}^2 \quad . \tag{9}$$

$$\mathrm{Var}[\mathrm{sgn}(f + X)] = 1 - f^2/a^2 \quad . \tag{10}$$

Consequently, the variance is minimal when $|f| = a$ or for a signal-to-noise amplitude ratio of unity. When the variance is minimal, recovery is optimal.

GAUSSIAN NOISE

Now, let the probability density be that of a Gaussian distribution (Rice, 1995)

$$\rho(x;\mu;s) = [1/\sigma\sqrt{(2\pi)}]e^{-\frac{1}{2}[(x-\mu)/\sigma]^2} \quad , \tag{11}$$

where μ is the mean and σ is the standard deviation. Because we require that the density be symmetric about $x = 0$, $\mu = 0$ and eq. (6) becomes

$$E[\mathrm{sgn}(f + X)] = \int_{-f}^{f} [1/\sigma\sqrt{(2\pi)}]e^{-\frac{1}{2}[x/\sigma]^2}dx \quad . \tag{12}$$

Make the change of variable $t = x/\sigma$. Then eq. (12) becomes

$$E[\mathrm{sgn}(f + X)] = [1/\sqrt{(2\pi)}] \int_{-f/\sigma}^{f/\sigma} e^{-\frac{1}{2}t^2}dt \quad . \tag{13}$$

Because the Gaussian is symmetric about zero, equation (13) can be written as

$$E[\mathrm{sgn}(f + X)] = [2/\sqrt{(2\pi)}] \int_{0}^{f/\sigma} e^{-\frac{1}{2}t^2}dt \quad . \tag{14}$$

Let

$$q = t/\sqrt{2} \quad , \tag{15}$$

$$dq = dt/\sqrt{2} \quad , \tag{16}$$

and

$$\tilde{f} = f/\sqrt{(2\sigma)} \quad . \tag{17}$$

Eq. (14) becomes

$$E[\mathrm{sgn}(f + X)] = (2/\sqrt{\pi}) \int_{0}^{\tilde{f}} e^{-q^2}dq \quad . \tag{18}$$

The RHS of eq. (18) is the Gauss error function $\mathrm{erf}(\tilde{f})$ (Kahn, 1990). Consequently, for zero-mean Gaussian noise,

$$E[\mathrm{sgn}(f + X)] = \mathrm{erf}[f/\sqrt{(2\sigma)}] \quad . \tag{19}$$

The error function is a sigmoid curve which is asymptotic at ± 1.

$\mathrm{erf}(\tilde{f})$ has a Maclaurin series representation:

$$\mathrm{erf}(\tilde{f}) = (2/\sqrt{\pi}) \sum_{n=0}^{\infty} [(-1)^n \tilde{f}^{2n+1}/n!(2n + 1)] \quad , \tag{20}$$

or

$$\mathrm{erf}(\tilde{f}) = (2/\sqrt{\pi})(\tilde{f} - (\tilde{f}^3/3) + (\tilde{f}^5/10) - (\tilde{f}^7/42) + (\tilde{f}^9/216) - \ldots) \quad . \tag{21}$$

Observe that $\mathrm{erf}(\tilde{f})$ is odd. That is,

$$\mathrm{erf}(-\tilde{f}) = -\mathrm{erf}(\tilde{f}) \quad . \tag{22}$$

The error function is shown in Fig. 1.

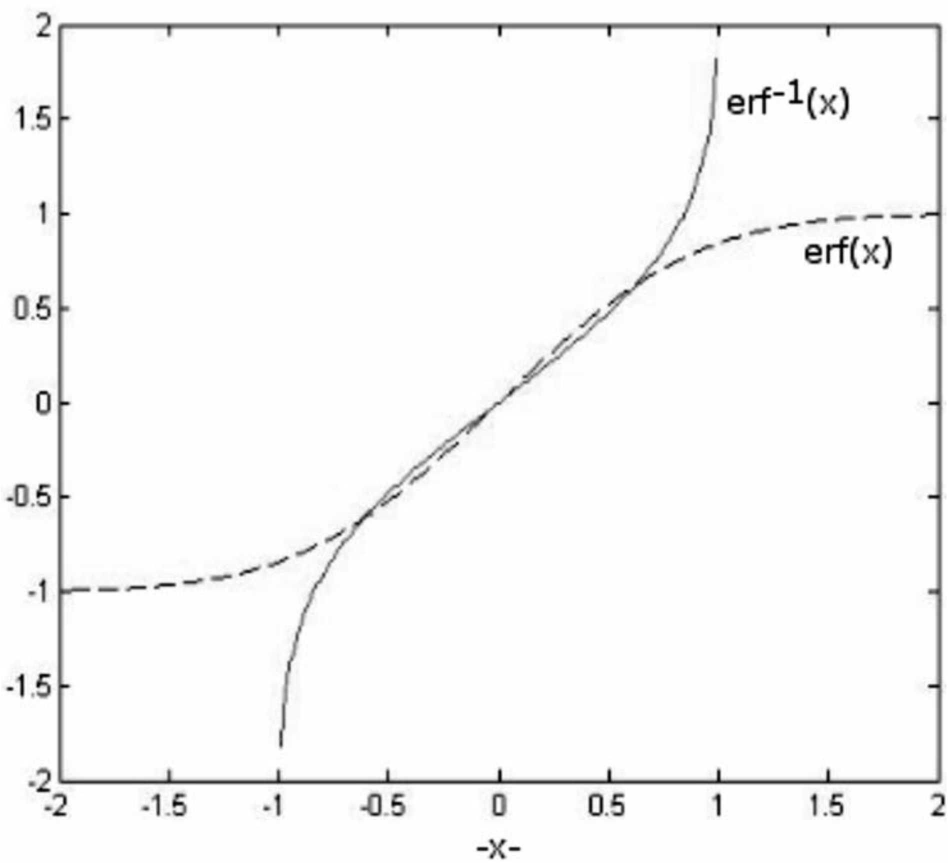

Fig. 1. The functions erf(x) and $\mathrm{erf}^{-1}(x)$.

AN ALTERNATIVE DERIVATION

There is an alternative way to derive these results. We present it as follows. Treat the signal as a differentiable function f(z). Then based on linearity we can write

$$(d/dz)E[\mathrm{sgn}(f(z) + X)] = E[(d/dz)\mathrm{sgn}(f(z) + X)] \quad , \tag{23}$$

$$(d/dz)\mathrm{sgn}(f(z) + X) = 2\delta(f(z) + X)(df/dz) \quad . \tag{24}$$

Consequently,

$$E[(d/dz)\mathrm{sgn}(f(z) + X)] = 2(df/dz) \int_{-\infty}^{\infty} \delta(f(z) + X)\rho(x)dx$$
$$= 2\rho(-f)(df/dz) \quad . \tag{25}$$

Using a zero-mean Gaussian density, eq. (25) becomes

$$E[(d/dz)\mathrm{sgn}(f(z) + X)] = [2/\sigma\sqrt{(2\pi)}]e^{-f^2/2\sigma^2}(df/dz) \quad , \tag{26}$$

or using eq. (23) and the fact that

$$(d/du)\mathrm{erf}(u) = (2/\sqrt{\pi})]e^{-u^2} \quad , \tag{27}$$

yields

$$(d/dz)E[\mathrm{sgn}(f(z) + X)] = (d/dz)\mathrm{erf}[f(z)/\sqrt{(2\sigma)}] \quad , \tag{28}$$

Thus,

$$E[\mathrm{sgn}(f(z) + X)] = \mathrm{erf}[f(z)/\sqrt{(2\sigma)}] + C \quad , \tag{29}$$

where C is some constant. For f(z) = 0, eq. (29) becomes

$$E[\mathrm{sgn}(X)] = C \quad , \tag{30}$$

or

$$C = 0 \quad , \tag{31}$$

which must be true for all values of f(z) and thus, eq. (29) is equivalent to eq.(19). We now show that the same type of analysis can be used for uniform noise. Let

$$\rho(-f) = 1/2a \quad . \tag{32}$$

Eq. (25) becomes

$$E[(d/dz)\mathrm{sgn}(f(z) + X)] = 2(1/2a)(df/dz) = (d/dz)(f/a) \quad . \tag{33}$$

Consequently, using eq. (23),

$$E[\mathrm{sgn}(f(z) + X)] = (f/a) + K \quad , \tag{34}$$

where K is some constant. For f(z) = 0, eq. (34) becomes

$$E[\mathrm{sgn}(X)] = K \ , \tag{35}$$

or

$$K = 0 \ , \tag{36}$$

which must be true for all values of f(z). Thus, eq. (34) is equivalent to eq. (8).

OPTIMAL SIGN-BIT RECOVERY FOR GAUSSIAN NOISE

We found that for uniform noise when S/N = 1, there is optimal sign-bit amplitude recovery. This result can be extended to Gaussian noise by using the Central Limit Theorem. As explained by Papoulis (1965) and Papoulis (1962), according to the Central Limit Theorem, given n independent random variables, X_i, we form the sum

$$X = X_1 + X_2 + \ldots + X_n \ . \tag{37}$$

This is a random variable with mean $\eta = \eta_1 + \ldots + \eta_n$ and variance $\sigma^2 = \sigma_1^2 + \ldots + \sigma_n^2$. Under certain general conditions, the Central Limit Theorem states that the distribution of x approaches a normal distribution with the same mean and variance as n increases.

The density $\rho(x)$ of the sum is given by the convolution

$$\rho(x) = \rho_1(x) * \rho_2(x) * \ldots * \rho_n(x) \ . \tag{38}$$

According to the Central Limit Theorem, the density $\rho(x)$ approaches under certain general conditions a normal curve as n increases. If the densities, $\rho_i(x)$ are reasonably concentrated near their respective mean, then the normal curve is a close approximation to $\rho(x)$ even for moderate values of n.

$$\rho(x) = [1/\sigma\sqrt{(2\pi)}]e^{-(x-\eta)^2/2\sigma^2} \ . \tag{39}$$

Papoulis (1965) and Papoulis (1962) give short examples for n = 2 and n = 3. The random variables are independent, identically distributed and uniformly distributed in the interval (0,1). For n = 1, $\rho(x)$ is a pulse. For n = 2, $\rho(x)$ is a triangle obtained by convolving a pulse with itself. For n = 3, $\rho(x)$ consists of three parabolic pieces obtained by convolving a triangle with a pulse. In the n = 3 case we come close to approximating a normal curve.

This result shows that the case of a uniform noise distribution as discussed by O'Brien et al. (1982), which results in an unbiased estimator of f(t) provided that S/N $\leq$ 1 , holds for the case of Gaussian noise as well. This is the case because for example for 3n recordings of single bit data we may establish by summing every three recordings a data set of n recordings of single bit data, which already are very close to a normal curve.

Therefore also for Gaussian noise there exists an optimal S/N ratio for amplitude recovery and this corresponds (for large n) to that being obtained for the case of uniform noise, which is S/N = 1.

PROCESSING THE SIGN-BIT AVERAGE

Using the inverse error function (Carlitz, 1963), we can process the sign-bit average to recover the signal exactly as follows

$$f = \sigma\sqrt{2}\,\mathrm{erf}^{-1}[E(\mathrm{sgn}(f + X))] \ . \tag{40}$$

The inverse error function has a Maclaurin series representation:

$$\mathrm{erf}^{-1}(z) = \sqrt{\pi}\,[(1/2)z + (1/24)\pi z^3 + (7/960)\pi^2 z^5 + (127/80640)\pi^3 z^7 + \ldots] \ . \tag{41}$$

The inverse error function is shown in Fig. 1.

COMPUTATIONAL TESTS

Using the MATLAB platform, simulations of sign-bit amplitude recovery were run with synthetic signal and noise. In the first test, shown in Fig. 2 we compare a signal, the damped sync function $\sin c(x/3)e^{-(x/15)^2}$, to the signal plus noise for both uniform noise and Gaussian noise. We chose this waveform because the sync function is one of the simplest waveforms to generate and although this signal is acausal and symmetric, sign-bit amplitude recovery is independent of the shape of the signal. In this example, the uniform noise magnitude is one (a = 1) and the standard deviation of the Gaussian noise is one (σ = 1). Observe that in both noise cases, the signal is corrupted by the noise.

The second test involved uniform noise with a magnitude of one (a = 1) and signal $[\sin c(x/3)e^{-(x/15)^2}]$ with a magnitude of one (i.e., S/N = 1). The purpose of this test is to demonstrate the sign-bit amplitude recovery phenomenon. Results of the first test are shown in Fig. 3. There are five traces, graphed from bottom to top as follows: signal, signal plus noise, sign of the signal plus noise, signal plus noise averaged over 200 iterations and sign of the signal plus noise averaged over 200 iterations. As shown in the latter two traces, amplitude recovery is clearly evident in the results.

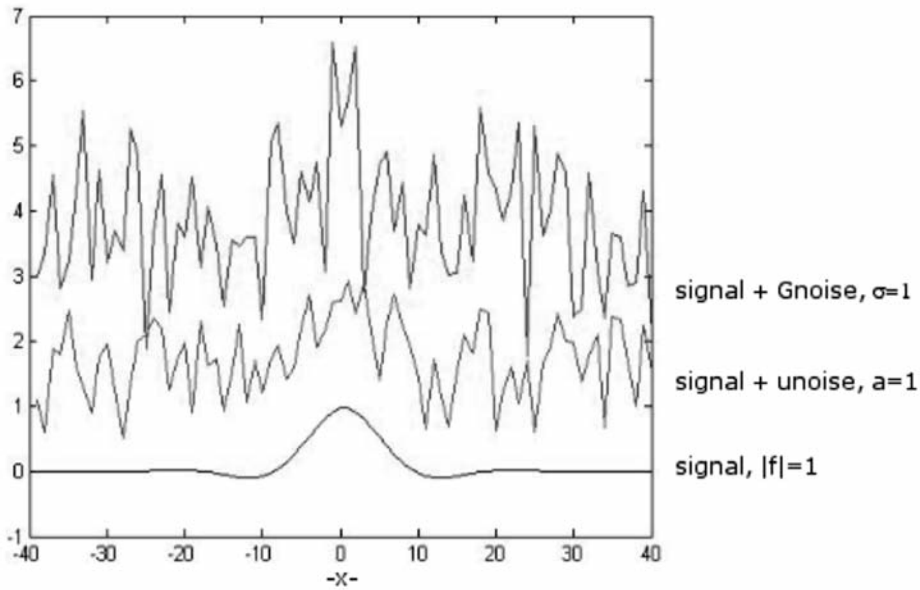

Fig. 2. Plot of a signal [sin c(x/3)$e^{-(x/15)^2}$], signal plus uniform noise (unoise) of magnitude one (a = 1), and signal plus Gaussian noise (Gnoise) with $\sigma = 1$.

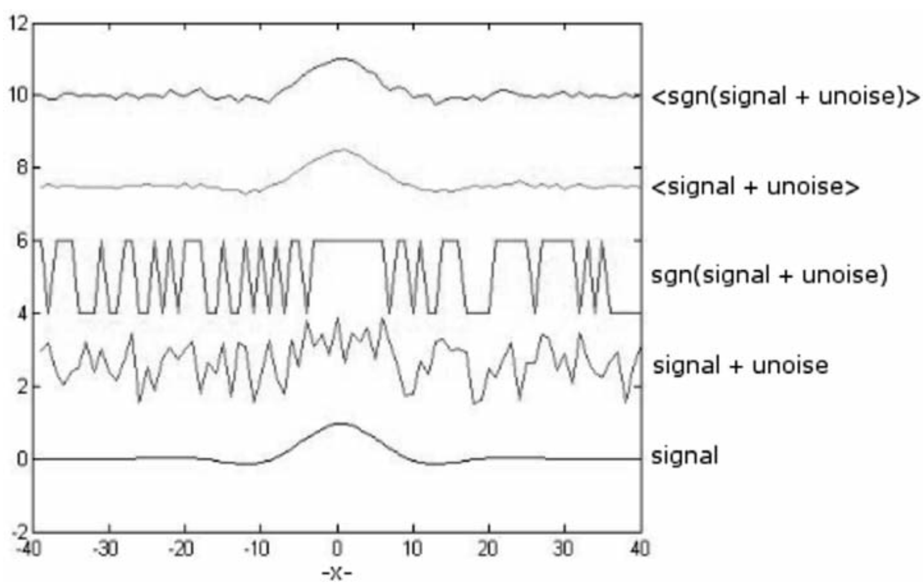

Fig. 3. A synthetic signal [sin c(x/3)$e^{-(x/15)^2}$] and the computer-generated uniform random noise used to examine sign-bit amplitude recovery. This test has noise of unit magnitude (a = 1) and a signal-to-noise ratio of one. Shown from bottom to top is the signal, the signal plus noise, the sign of the signal plus noise, the average over 200 iterations of signal plus noise, and the average over 200 iterations of the sign of signal plus noise.

As shown in Fig. 4, the third test compares the unit magnitude signal [sinc(x/3)$e^{-(x/15)^2}$] to the sign-bit average over 200 iterations for signal and unit magnitude, uniform noise (a = 1) and to the sign-bit average over 200 iterations for signal and Gaussian noise (σ = 1). Observe the difference in the waveforms. The signal is clearly underrepresented by Gaussian noise sign-bit averaging. This test was repeated several times with the same results. According to theory, Gaussian noise sign-bit averaging produces an error function of the signal, not the original signal.

In the fourth test the third test is repeated but the waveform with Gaussian noise is inverted with the inverse error function as given by eq. (40). The results of this test are shown in Fig. 5. In this test we see more correlation between the waveforms, which is consistent with theory. As with the prior test, these results have repeatability.

In sign-bit amplitude recovery from uniform noise, clipping occurs when the signal is greater than the noise. For Gaussian noise, clipping occurs as the error function becomes asymptotic. In the fifth test, shown in Fig. 6, we compare the hard clipping associated with uniform noise to the soft clipping associated with Gaussian noise by using a signal with magnitude two [sinc(x/3)$e^{-(x/15)^2}$], unit magnitude uniform noise (a = 1), and unit standard deviation for the Gaussian noise (σ = 1).

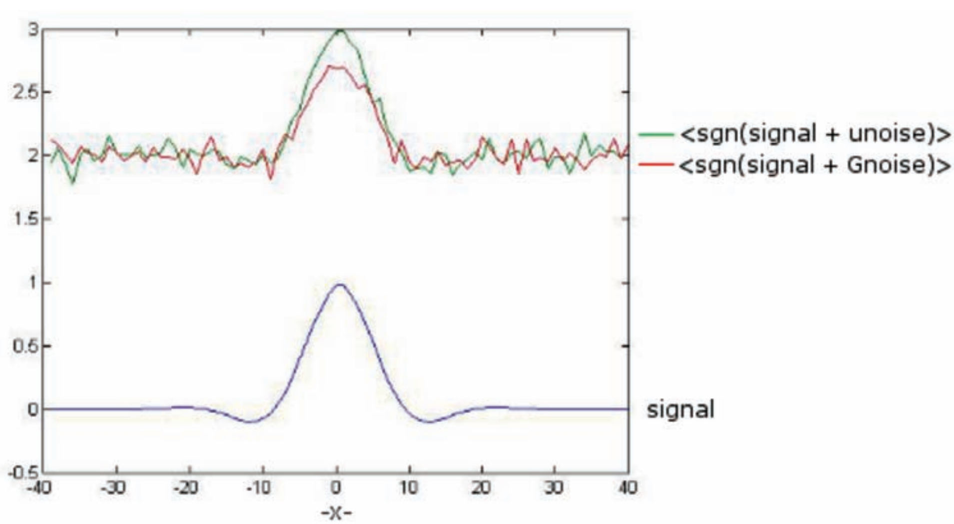

Fig. 4. At bottom: plot of signal [sin c(x/3)$e^{-(x/15)^2}$]. At top: overlay of the average over 200 iterations of the sign of signal plus Gaussian noise (Gnoise) with σ = 1 and the average over 200 iterations of the sign of signal plus unit magnitude (a = 1), uniform noise (unoise).

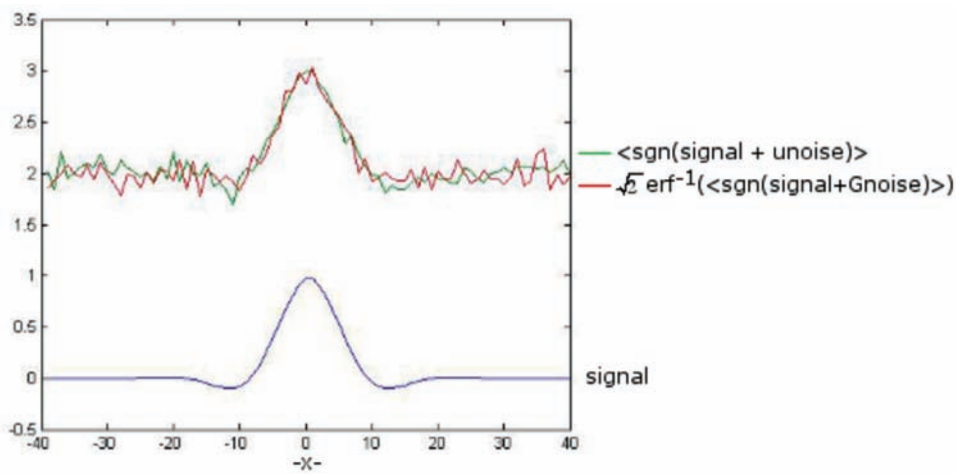

Fig. 5. At bottom: plot of signal [sin c(x/3)$e^{-(x/15)^2}$]. At top: overlay of the $\sqrt{2}$ times the inverse error function of the average over 200 iterations of the sign of signal plus Gaussian noise (Gnoise) with $\sigma = 1$ and the average over 200 iterations of the sign of signal plus unit magnitude (a = 1), uniform noise (unoise).

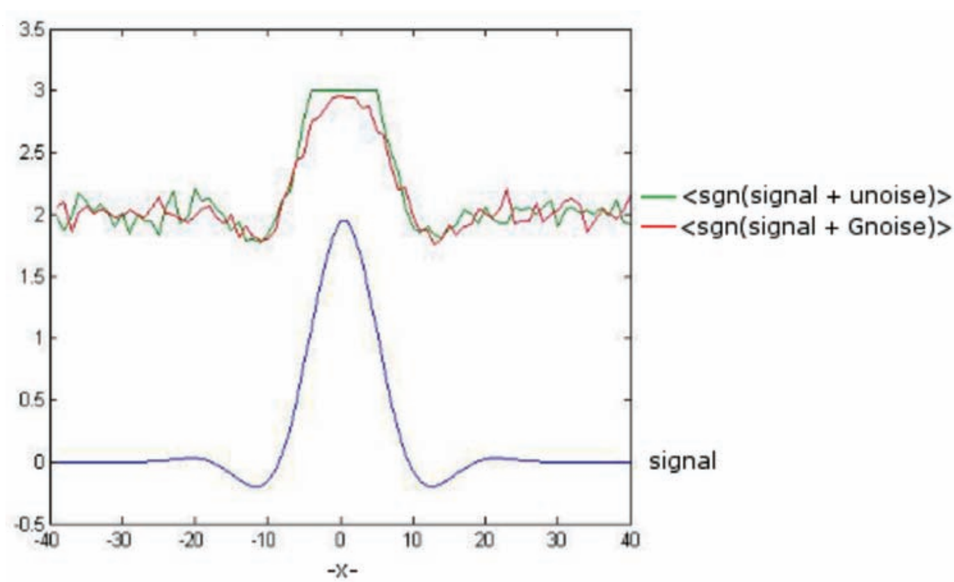

Fig. 6. This plot compares the hard clipping produced by uniform noise to the soft clipping produced by Gaussian noise in sign-bit averaging. At bottom: plot of signal [sin c(x/3)$e^{-(x/15)^2}$]. At top: overlay of the average over 200 iterations of the sign of signal plus Gaussian noise (Gnoise) with $\sigma = 1$ and the average over 200 iterations of the sign of signal plus unit magnitude (a = 1), uniform noise (unoise).

CONCLUSIONS

In this paper we derive sign-bit averaging for Gaussian noise. We deduce that sign-bit amplitude recovery does not directly occur for Gaussian noise. Rather, the sign-bit average produces the Gauss error function which has to be inverted in order to recover the signal. Because the error function can produce facsimiles of the signal, the behavior of this process can be mistaken for the direct signal recovery that occurs for uniform noise. We also point out that because Gaussian noise causes soft clipping, clipping can also be misidentified as signal in this case. These results are tested computationally using synthetic data and the tests agree with theory. We add that like the case for uniform noise which has optimal recovery for a unit signal-to-noise ratio, there is an optimal signal recovery for Gaussian noise and it is identical to the case for uniform noise.

This paper offers the possibility that if the data are recorded in the presence of Gaussian noise, sign-bit amplitude recovery can be improved. However, experiments would be required to determine when Gaussian noise dominates in the field for new recordings or to determine if and where Gaussian noise dominates existing sign-bit recordings. This work only offers a preliminary mathematical solution to the problem, but it is hoped that it may serve as an introduction to a more rigorous development. Finally, this paper does illustrate a difference between sign-bit amplitude recovery in uniform and Gaussian noises. The approach used in this work is open to the application of other possible noises for future study.

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