ANELASTIC (POROVISCOELASTIC) MEDIUM - THE S_H - WAVE PROBLEM

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ABSTRACT

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When considering the problem of extending seismic wave propagation in an elastic medium to a poroviscoelastic medium, replacing real quantities by complex equivalents has been the accepted way to proceed. Given the number of works dealing with, what could be called the inadequacy of this method of approach, another line of reasoning might be in order. Starting with Biot's equations for a poroviscoelastic medium, employing a simplification route, results in the S_H (modified) potential related to the vector equation of motion. Biot's theoretical development of wave propagation in a medium comprised of a fluid within a porous solid may be overly complicated for the pursuit of an alternate methodology for addressing this problem in its most basic form. As a consequence, the *telegraph* equation might be a more modest, yet informative analogue to consider, as it is a well studied problem from mathematical and physical perspectives. In what follows an S_H potential wave equation is considered with attenuation introduced in a manner similar to that inherent in the telegraph equation. Additionally, the *difficult* situation discussed by Krebes and Daley will again be revisited, as it might be rationalized that a 1 - 2% modification of a real quantity such as velocity produces imperceptible effects in, say a reflection coefficient, while the same amount of perturbation introduced to make velocity a complex quantity results in significant dissimilarities between nearly similar initial input data. This is difficult to comprehend and seemingly at least as problematic to explain.

KEY WORDS: anelasticity, SH wave propagation, reflection and transmission coefficients, Riemann sheet, multivalued complex functions.

INTRODUCTION

It might be superfluous to begin with the phrase "In the recent literature the topic of wave propagation in poroviscoelastic media has received much attention", as it has been an ongoing area of investigation for a number of decades. A series of papers by Morozov (2009a, 2009b, 2010, 2011), Morozov

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and Ma (2009) and others such as Lines et al. (2008), Ruud (2006) and Krebes and Daley (2007) have recently considered this problem. There are a significant number of other relevant citations within all of the above including the standard texts of Aki and Richards (1980, 2002) and Carcione (2007). Some of the above will be specifically referred to here.

As mentioned in the Abstract an analogy of the telegraph equation can be used to pursue a solution method. The reason for this is to not have to assume that the correspondence principle, Carcione (2007), is valid, which results in not having to require, in going from elastic to (poro)viscoelastic media, that real media parameters be replaced by complex valued equivalents. The telegraph equation is a problem encountered in almost all advanced undergraduate mathematical or mathematical physics texts. One of its forms for an infinite line is

$$(\partial^2 u/\partial x^2) - \alpha(\partial u/\partial t) - \gamma(\partial^2 u/\partial t^2) = 0, \quad t > 0 \ . \tag{1}$$

$$u(x,t)|_{t=0} = (\partial u/\partial t)(x,t)|_{t=0} = 0, \quad u(0,t) = f(t), \quad u(x,t)|_{x\to\infty} = 0. \quad (2)^*$$

This problem is often used as an example in presenting Laplace transform theory (Hildebrand, 1962) and can be solved using the Laplace transform tables on pp. 1020-1029 in Abramowitz and Stegun (1980). The two constants, are composed of the quantities R - resistance , C - capacitance and L - inductance (all per unit length) which are real positive quantities and $\alpha = RC$, $\gamma = LC$. As mentioned, a similar higher spatial dimension form of this equation type, specific to seismic wave propagation, may be obtained from Biot's equations [Biot (1956a, 1956b, 1956c)]** for wave propagation in a poroviscoelastic medium for the S_H potential wave equation, or equivalently the telegraph equation may be generalized to more spatial dimensions, as

$$\mu\nabla^2\psi - b(\partial\psi/\partial t) - \rho(\partial^2\psi/\partial t^2) = F(\mathbf{x},t) = \delta(\mathbf{x})f(t) \ , \tag{3}$$

with zero initial conditions

$$\psi(\mathbf{x},t)|_{t=0} = (\partial\psi/\partial t)(\mathbf{x},t)|_{t=0} = 0 \ , \tag{4}$$

where $\psi(\mathbf{x},t)$ is the S_H wave potential whose corresponding polarization vector is oriented perpendicular to the plane of incidence in what has been assumed to

* Most often the boundary condition is given at some finite length l as $u(x,t)|_{x\to l} = 0$ (line open) and then let $l \to \infty$. $du(x,t)/dx|_{x\to l} = 0$ (line grounded).

** Frenkel, circa 1935, published a work in Russian on this topic that has subsequently been translated to English. This work has some inconsistencies. Most present day citations in this area of research refer to Biot.

be a homogeneous medium. The quantity μ is Lamé's rigidity parameter, ρ - volume density and b - a dimensionally correct constant. In Biot's theory, the quantity b is defined as the mobility ratio in terms of real parameters as $b = \phi^2\eta/k$, where ϕ - porosity, η - viscosity and k - permeability. All of the preceding values are real and positive quantities.

This approach is consistent with what was demonstrated by Razavy (2005), in that attenuation in the elastic wave equation is described not by modifying the Lamé parameters λ and μ to make them complex quantities, but rather by adding an external dissipation force to Lagrange's equations from which the equations of motion for (an)elastic media are obtained using the classical approach.

What has been assumed here is that only medium types that display attenuation visible on seismic records are those composed of a solid matrix with a fluid of any type occupying the porous part of the medium. This would exclude metamorphic formations such as a serpentine layer imbedded in a basalt structure, as there is little or no attenuation associated with this (in the strictest theoretical or geological sense).

PLANE WAVE THEORY

It is probably useful to first consider a two dimensional plane wave solution for eq. (3) of the form

$$\psi(x,z,t) = A\exp[-i\omega t + i\omega px + i\omega qz] \quad , \qquad (5)$$

for some nonzero amplitude A. Introducing (5) into the source free form of (3) results in

$$[\mu(i\omega p)^2 + \mu(i\omega q)^2 + i\omega b - \rho(i\omega)^2]A = 0 \quad , \qquad (6)$$

or as $A = 0$, then

$$\begin{aligned} p^2 + q^2 - (ib/\omega\rho\beta^2) - (1/\beta^2) &= 0 \\ q^2 = (1/\beta^2)[1 + (ib/\omega\rho)] - p^2 & \end{aligned} \quad , \qquad (7)$$

from which it follows that

$$q = \{(1/\beta^2)[1 + (i/\omega\rho/b)] - p^2\}^{1/2} \quad . \qquad (8)$$

ω is only here as a mathematical convenience in order to make the quantity dimensionless. If one starts from a finite difference solution $\omega \rightarrow (\Delta t/2)^{-1}$. Q must be determined empirically using other methods. However, k - effective permeability and η - effective viscosity both may be frequency dependent. Usually the ratio b is taken as the frequency dependent parameter, if this type of dependence is wanted, or $\bar{b} \rightarrow \omega b$, so that $Q = \rho/\bar{b}$.

Upon comparison of (8) with the vertical slowness defined in other works indicates that the dimensionless attenuation or quality factor Q is defined as $Q = \omega\rho/b$, and $p = p_r + ip$, with p_r and p_i being real positive quantities so that p is required to lie in the first quadrant of the complex p-plane to satisfy radiation conditions.

SH PLANE WAVE REFLECTION AND TRANSMISSION COEFFICIENTS

Before considering more complex solution methods related to this problem it is useful to begin with the plane wave reflection and transmission coefficients at an interface between two poroviscoelastic media. Consider two (1 $\rightarrow$ *upper* and 2 $\rightarrow$ *lower*) poroviscoelastic media separated by an interface in the (x,z) plane at z = 0 with z chosen to be positive downwards. Media parameters are $\beta_k^2 = \mu_k/\rho_k$, b_k, and Q_k. The incident, reflected and transmitted plane wave potentials may be written as

$$\psi_{inc} = A_{inc}\exp[-i\omega t + i\omega px + i\omega q_1 z]$$

$$\psi_{ref} = A_{ref}\exp[-i\omega t + i\omega px - i\omega q_1 z] \quad . \tag{9}$$

$$\psi_{trn} = A_{trn}\exp[-i\omega t + i\omega px + i\omega q_2 z]$$

It should be mentioned here that the reflection and transmission coefficients are the same, whether particle displacement or potentials are used. They are related by way of $\psi = \nabla\cdot\mathbf{u}$, where $\mathbf{u} = u\mathbf{e}_\phi$ and $\mathbf{e}_\phi$ is a unit vector perpendicular to the plane of incidence. With $A_{ref}/A_{inc} = R_{11}^{SH}$ being the reflection coefficient and $A_{trn}/A_{inc} = R_{12}^{SH}$ the transmission coefficient the continuity of displacement and shear stress at a plane interface between the two poroviscoelastic media using plane waves require that

$$R_{11}^{SH} - R_{12}^{SH} = -1 \quad , \tag{10}$$

and

$$R_{11}^{SH}\mu_1 q_1 + R_{12}^{SH}\mu_2 q_2 = \mu_1 q_1 \quad . \tag{11}$$

Defining D to be

$$D = \mu_1 q_1 + \mu_2 q_2 \quad , \tag{12}$$

results in the expressions for the reflection and the transmission coefficients for incidence from the upper medium to be

$$R_{11}^{SH}(\mu_1 q_1 - \mu_2 q_2)/D \quad , \qquad R_{12}^{SH} = 2\mu_1 q_1/D \quad , \tag{13}$$

where

$$q_k = [(1/\beta_k^2) - p^2]^{1/2} \rightarrow q_k = \{(1/\beta_k^2)[1+(i/Q_k)] - p^2\}^{1/2} \; (k = 1,2) \; . \tag{14}$$

As before, $p = p_r + ip_i$ $(p_r \geq 0,\ p_i \geq 0)$.

From the paper by Morozov (2011) the medium parameters used here for SH reflection are similar to what was used in that paper for the acoustic wave case. As Q is infinite in the upper medium, this indicates that the upper medium is elastic. As a consequence, the values of $p = p_0$ corresponding to plane wave incidence from medium 1 for the range of incident angles $(0 \leq \theta \leq \pi/2)$ is $(0 \leq p_0 \leq p_1)$, where p_1 is located on the real p-axis (Fig. 1). The lower medium is assumed to be poroviscoelastic with the relation between the shear wave velocities in the two media given by $\beta_1 < \beta_2$, $Q_1 < Q_2$.

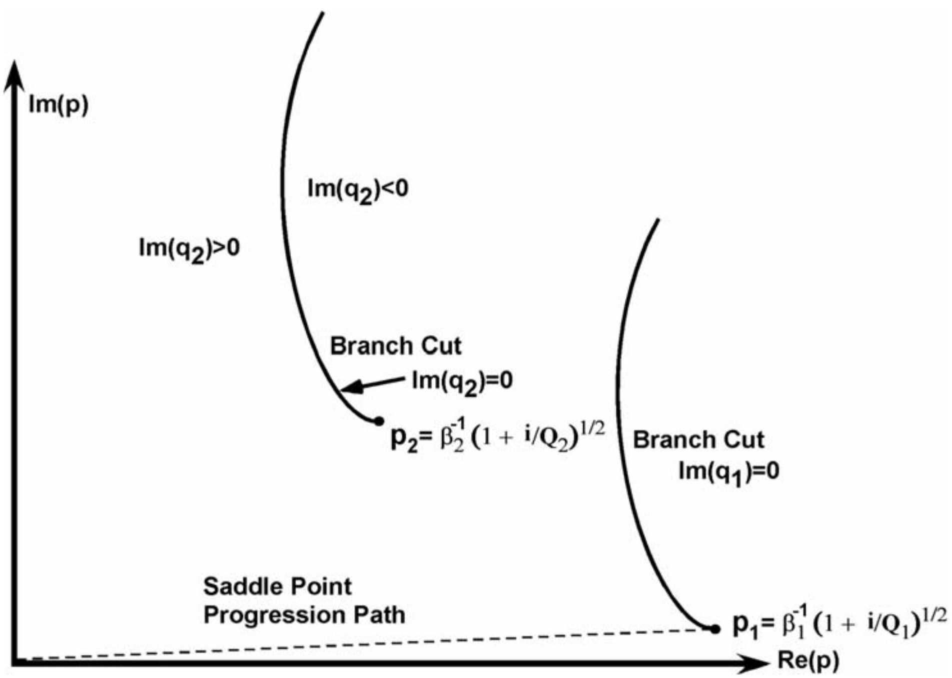

Fig. 1. This schematic shows the saddle point path for the zero - order saddle point approximation for the SH wave equation for shear wave reflection from the plane interface between the two media. The parameter values in Table 1 indicate that the saddle point path should lie along the real p-axis. It has been moved slightly into the first quadrant for viewing convenience.

Two plots are presented for values of Q = 30 and Q = 5 in medium 2 in Figs. 2 and 3. Each of these figures consists of an upper and lower panel. The upper panel contains the amplitude plotted against the real part of p_0 while the bottom panel is the phase versus Re(p_0). Both the poroviscoelastic coefficients and the reference elastic case ($Q_1 = Q_2 = \infty$) amplitudes and phases are shown in the figures. For completeness, the transmission coefficients for the two cases described in Table 1 are shown in Figs. 4 and 5. In all figures, the anelastic case is plotted in black and the elastic case is in gray with an "e" indicator.

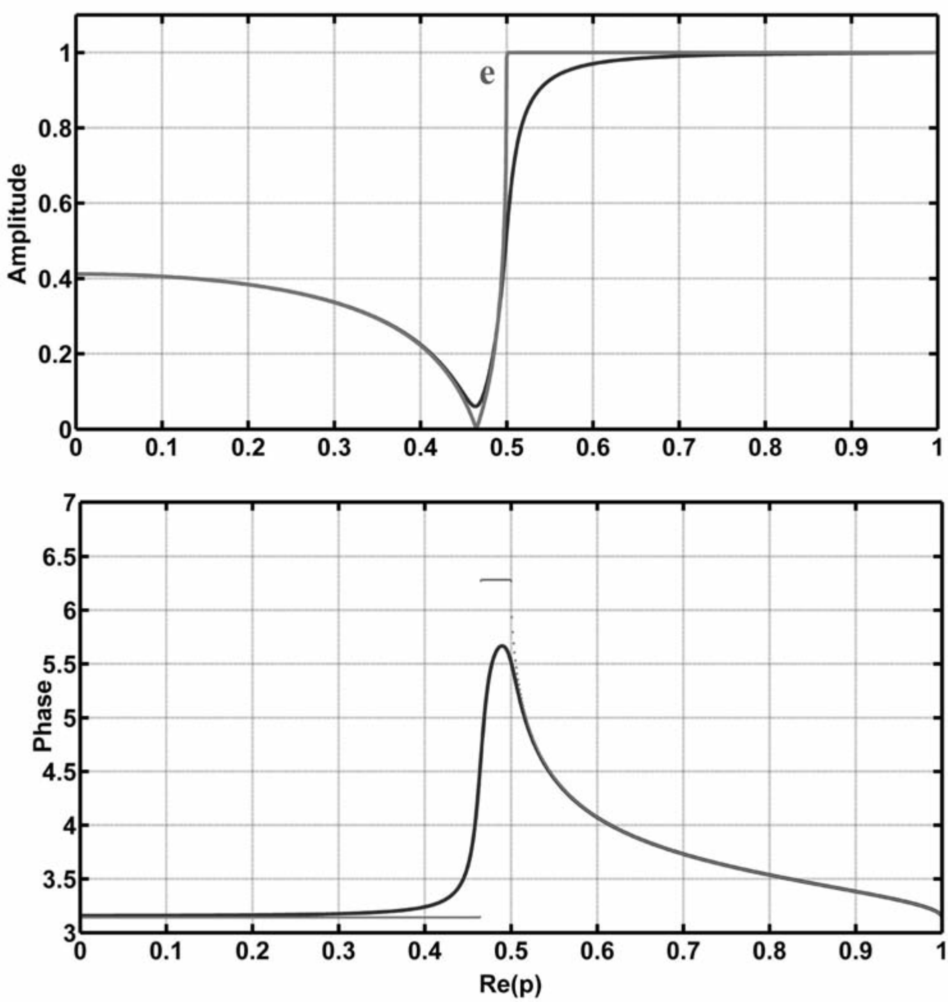

Fig. 2. The R_{11}^{SH} reflection coefficient at an interface between two media. The upper (1) medium is elastic while the lower (2) medium is poroviscoelastic with $Q_2 = 30$. Medium parameters are given in Table 1.

Table 1. Medium parameters taken from Morozov (2011). The upper medium is elastic.

	Shear Wave Velocity (km/s)	Density (g/cm^3)	Q
Upper (1) Medium	1.0	1.0	∞
Lower (2) Medium	2.0	1.2	30/5

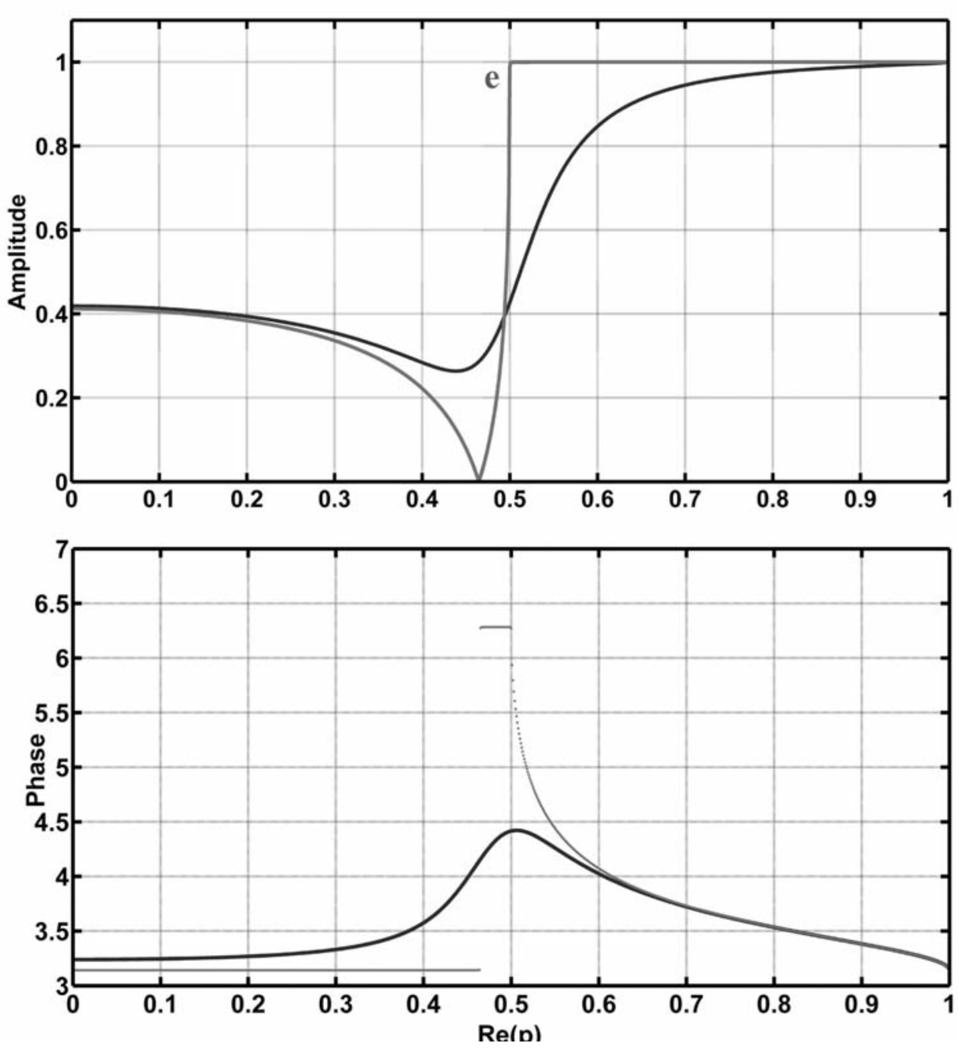

Fig. 3. The R_{11}^{SH} reflection coefficient at an interface between two media. The upper (1) medium is elastic while the lower (2) medium is poroviscoelastic with $Q_2 = 5$. Medium parameters are given in Table 1.

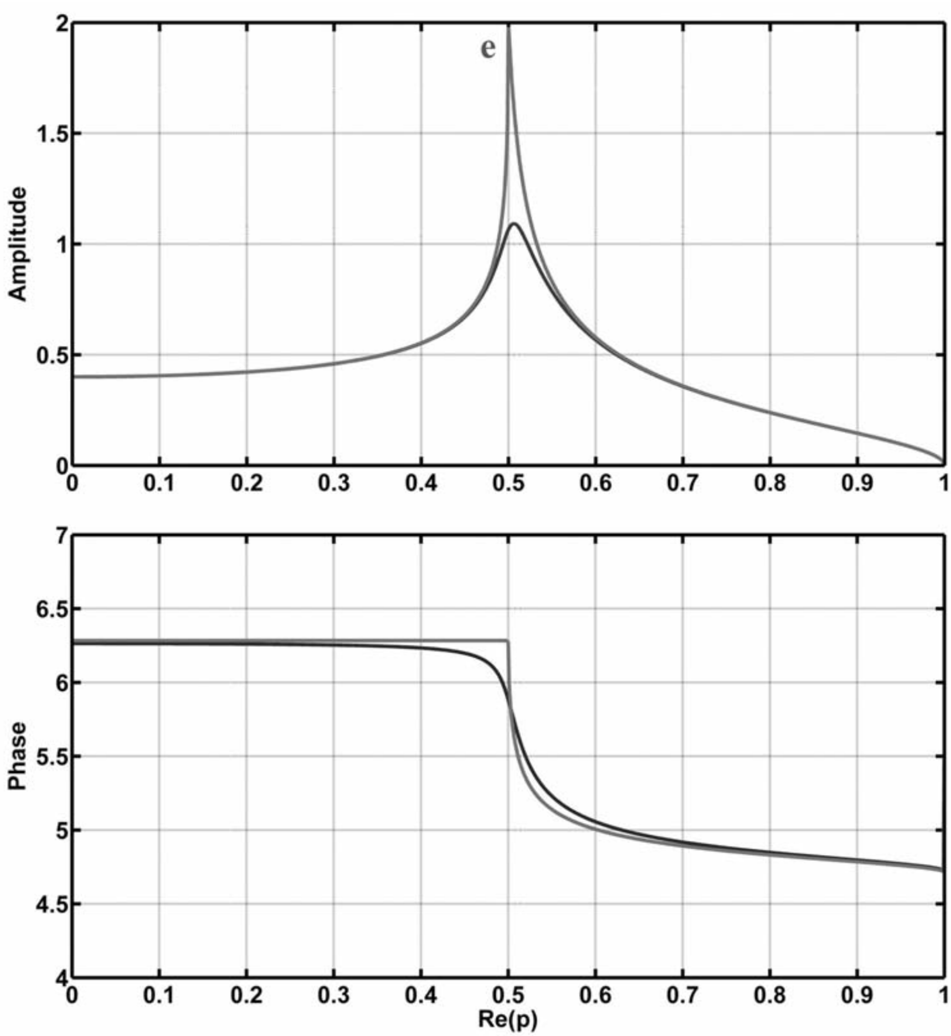

Fig. 4. The R_{12}^{SH} transmission coefficient at an interface between two media. The upper (1) medium is elastic while the lower (2) medium is poroviscoelastic with $Q_2 = 30$. Medium parameters are given in Table 1.

BRANCH POINTS AND RIEMANN SHEETS - THE DIFFICULT CASE

The *parameter* contrast at the interface between the two poroviscoelastic media for the difficult case (Fig. 6) is $\beta_2 > \beta_1$ and $Q_2 > Q_1$ where the incident/reflection (upper) medium is denoted as 1 and the medium of transmission (lower) as 2. As has been shown in many previous works on this problem, the location of the saddle point for all source receiver offsets $(0 \leq r < \infty)$ lies along the straight line from the origin of the p-plane to the point $p_1 = (1/\beta_1)[1 + (i/Q_1)]^{1/2}$.

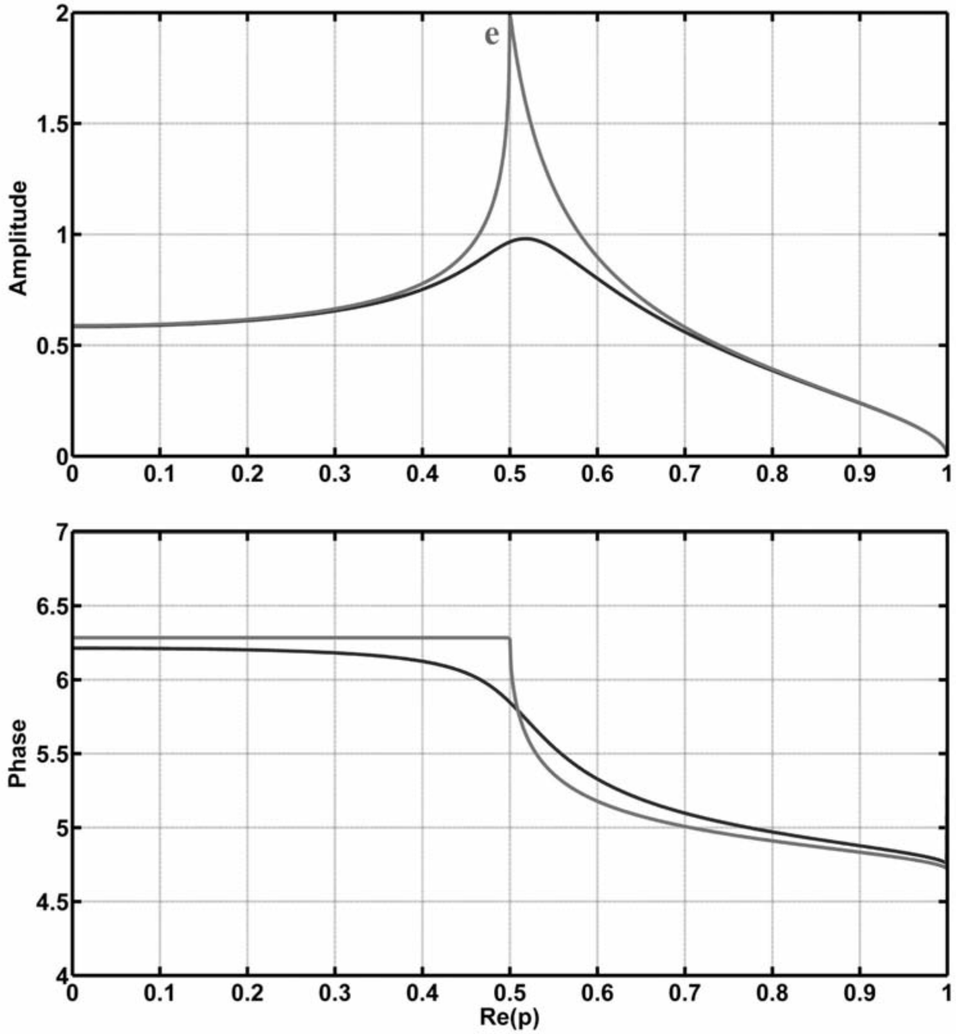

Fig. 5. The R_{12}^{SH} transmission coefficient at an interface between two media. The upper (1) medium is elastic while the lower (2) medium is poroviscoelastic with $Q_2 = 5$. Medium parameters are given in Table 1.

Consider a region **R** of the complex p-plane consisting of the first quarter of this space, inclusive of the real positive axis. A function q(p) is said to be analytic in this region if it has a finite derivative at each point in **R** and if q(p) is single valued in **R**. If q(p) is analytic at some point p ∈ **R** then the derivative of q(p) is continuous at p. In fact, it doesn't matter from which direction the point p ∈ **R** is approached. The derivative is always continuous. Further, the derivatives of q(p) are continuous for all orders. Any point p ∈ **R** that satisfies the above criteria is referred to an *interior* point of **R**.

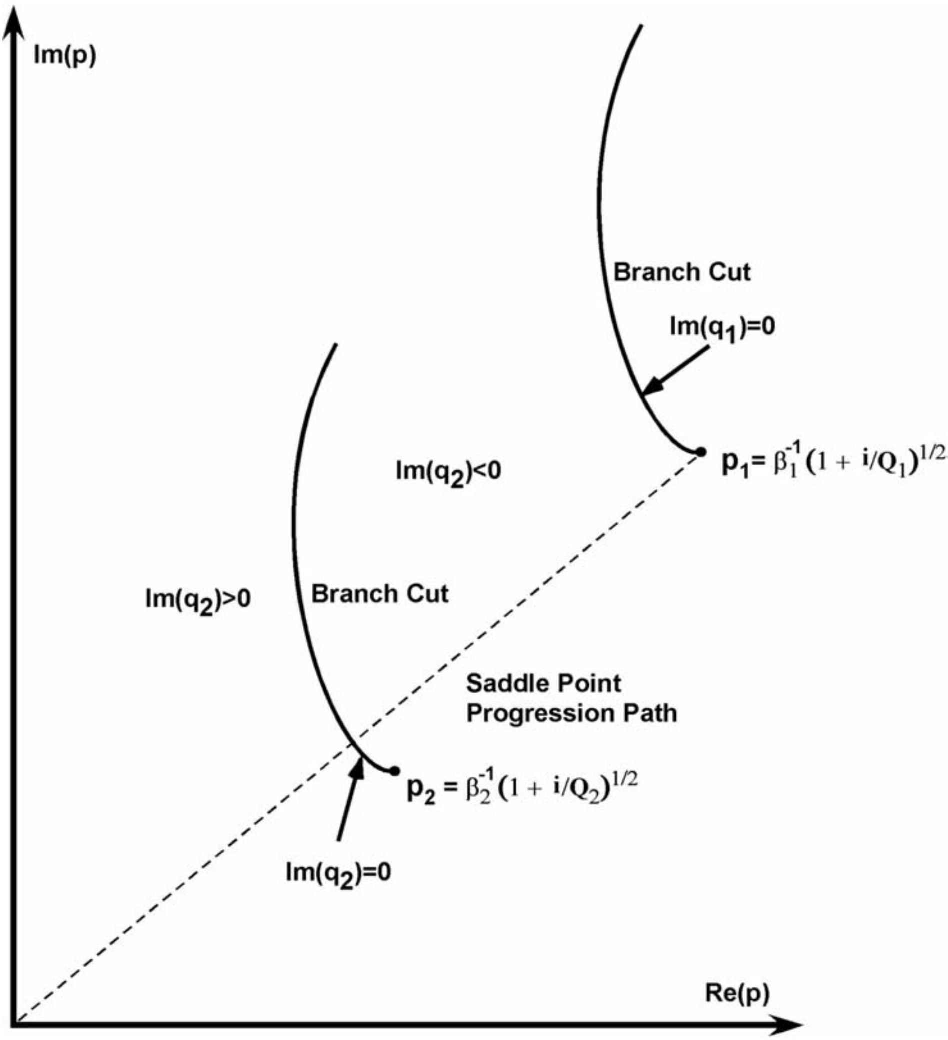

Fig. 6. This schematic shows the saddle point path for the zero - order saddle point approximation for the SH wave equation for shear wave reflection from the plane interface between the two media. The parameter values in Table 2 indicate that the saddle point path should lie along the line from the origin to p_1. This path requires that the branch cut associated with $p = p_2$ be crossed. The radical q_2 is required to remain on the first Riemann sheet as a result of arguments presented in the text.

If any region of **R** exists such that q(p) or a *branch* of q(p) is analytic in **R** then we still speak of q(p) as "an analytic function". Points at which q(p) is not analytic are called *singular points* or *singularities* of the function. Also, a point at which a branch of the multivalued function is not analytic is called a singular point of q(p). Thus, a singular point is not an interior point. To minimize the complexity of the above in further discussions of this matter where the theory of Riemann sheets is required to be formally introduced, let it be stated that when dealing with a multivalued function, possessing at least one branch point, it is usually desirable to specify one particular branch of the function and *artificially* prevent the possibility of a transition from that branch to another. In other words, given some path in **R**, which crosses a branch cut (defined below) the function q(p) is analytic along this path and by definition is continuous and possesses continuous derivatives along this path.

Table 2. Medium parameters for the difficult case, taken from Table 1. The two models here are for $Q_2 = 30$ and 50 with $Q_1 = 20$ for both.

	Shear Wave Velocity (km/s)	Density (g/cm^3)	Q
Upper (1) Medium	1.0	1.0	20
Lower (2) Medium	2.0	1.2	30/50

The complex valued radical q_2, defined in the complex p-plane, may be written as

$$q_2 = (p_2^2 - p^2)^{1/2} = (p_2 + p)^{1/2}(p_2 - p)^{1/2} \quad , \tag{15}$$

and has branch points at $p = p_2$ and $p = -p_2$. As only p values in the upper right (first) quadrant are of interest in what is considered here, the branch point at $p = p_2$ is of necessary concern. Before proceeding it should be noted that both branch points have *related* branch points at $p = \pm\infty$ in the upper and lower half planes (manifolds) of the complex p-plane. Any path from $p = p_2$ to some point at infinity may be taken as a branch cut. Here, a more specific requirement will be invoked: *The branch cut from $p = p_2$ to $p = +\infty$ will coincide with the path defined by the relationship $Im(q_2) = 0$, which defines the branch cut associated with the branch point $p = p_2$.*

It may be useful now to consider a reference function that is analytic and continuous in the first quadrant of the complex p-plane:

$$q_2^2 = (p_2^2 - p^2) = |q_2^2| e^{i\phi} \quad , \tag{16}$$

where $|q_2^2|$ is the modulus of this generally complex quantity and $\phi = \tan^{-1}[Im(q_2^2)/Re(q_2^2)]$ is its phase. Introduce the related functions

$$(q_2^2)_m = |q_2^2| e^{i\phi + i2m\pi} \quad , \tag{17}$$

under the assumption that no other singularities are present in the vicinity of the existence of this function. The 2π rotations are taken in a counterclockwise sense around the branch point at $p = p_2$ and the first three m functions in the series are

$$\begin{aligned} (q_2^2)_0 &= |q_2^2| e^{i\phi} \\ (q_2^2)_1 &= |q_2^2| e^{i\phi + i2\pi} \quad . \\ (q_2^2)_2 &= |q_2^2| e^{i\phi + i4\pi} \end{aligned} \tag{18}$$

With this it follows that the first three m functions associated with q_2 are

$$\begin{aligned} (q_2)_0 &= |q_2| e^{i\phi/2} \\ (q_2)_1 &= |q_2| e^{(i\phi + i2\pi)/2} = -|q_2| e^{i\phi/2} \quad . \\ (q_2)_2 &= |q_2| e^{(i\phi + i4\pi)/2} = |q_2| e^{i\phi/2} = (q_2)_0 \end{aligned} \tag{19}$$

The above 3 equations are standard in complex variable analysis and possibly unnecessary to repeat here. However, they do explain the fact that q_2 may be defined on two Riemann sheets in the upper right hand quadrant of the complex p-plane. It may be further noticed that in the first quadrant the quantity q_2 may be forced to stay on the first sheet if the complex conjugate value of q_2 is used after crossing the branch cut, as pointed out in Krebes and Daley (2007). As q_2^2 is a continuous analytic function in the first quadrant of the complex p-plane, it is reasonable, based on previous arguments to require that q_2 have similar properties.

All of this section may be written may be written more generally for the specific problem considered. Let $h(p) = (p_2^2 - p^2)$ be an analytic, continuous function defined in the first quadrant of the complex p-plane. Let the definition of a related function be given as $[h(p)]^{1/n} = (p_2^2 - p^2)^{1/n} = (p_2^2 - p^2)^{p/q}$, where n, p and q are integers and p and q have no common divisors except unity. For this problem it will further assumed that n, p and q are all positive.

Standard complex functions methods have

$$[h(p)]^{1/n} = |h(p)|^{1/n} \exp[(i\phi/n) + (i2\pi N/n) \text{ for N a positive integer.} \tag{20}$$

Any source of ambiguity in the above equation, resulting in $[h(p)]^{1/n}$ being a multivalued function is not due to the amplitude, $|h(p)|^{1/n}$, but rather to the fact that there are infinitely many choices for the phase, with the principle value defined with phase $\exp[i\phi/n]$ and $\phi = \tan^{-1}\{\mathrm{Im}[h(p)]/\mathrm{Re}[h(p)]\}$.

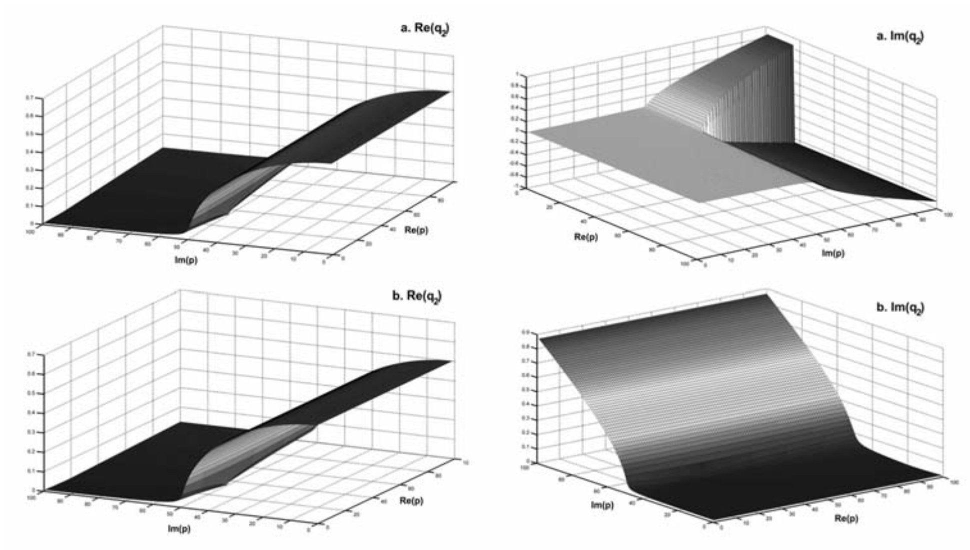

Fig. 7. The real and imaginary values of the multivalued function $q_2 = (p_2^2 - p^2)^{1/2}$ on the two possible Riemann sheets. For computational purposes sheet (b) where both real and imaginary parts of q_2 are continuous is used. Close examination of the individual figures reveals the position of the point p_2.

What is required here is that $[h(p)]^{1/n}$ is also continuous in the first quadrant of the p-plane. Let $[h(p_2)]^{1/n}$ be the value of $[h(p)]^{1/n}$ at p_2. Construct a circle of finite size (radius - ε) centered at p_2. It will be assumed that no other singularity of $[h(p)]^{1/n}$ lies within this circle. Choose another point on this circle at say $p = \hat{p}$, $\mathrm{Re}(\hat{p}) > \mathrm{Re}(p_2)$, so that the function value here is $[h(\hat{p})]^{1/n}$. Proceed along the circle in a counterclockwise direction until the point $[h(\hat{p})]^{1/n}$ is again reached. At this point $[h(\hat{p})]^{1/n}$ will return to its initial value or it will not. If one keeps this process up, circling the point p_2 a total of N times, where the value of $[h(\hat{p})]^{1/n}$ returns to its original value, it may be said that in the limit as $\varepsilon \rightarrow 0$, p_2 is a branch point of order $(N-1)$. For the case being considered here, $[h(p)]^{1/2} = (p_2^2 - p^2)^{1/2}$, $N = 2$. This may be observed in eq. (19) (Fig. 7).

Similar to the previous section, reflection and transmission coefficients will be presented for the model described in Table 2. Again the anelastic case is shown in black and the elastic case in gray. The reflection coefficients are shown in Figs. 8 and 9, with the related transmission coefficients given in Figs. 10 and 11.

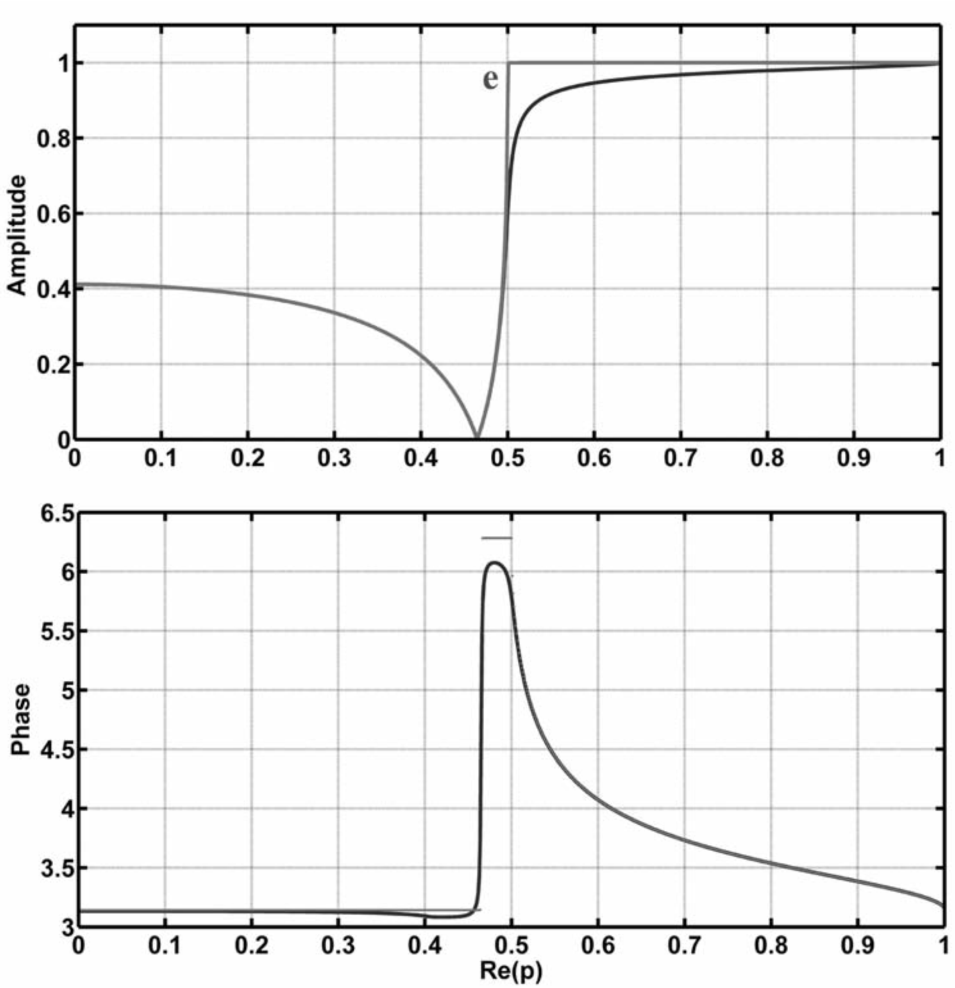

Fig. 8. The R_{11}^{SH} reflection coefficient at an interface between two medium. Both the upper (1) medium and the lower (2) media are anelastic with $Q_1 = 20$ and $Q_2 = 50$. Medium parameters are given in Table 2.

CONCLUSIONS

A method for dealing with wave propagation in attenuating media was investigated using equations that originated with Biot and earlier with Frenkel. Most often, if it is known that at some point in the solution, velocities may become complex quantities, they are introduced at the onset as complex quantities. Here, all initial parameters related to the problem are assumed to be real. At some point in the solution, a combination of these quantities may become a complex quantity. However, this evolution should be related to the theory used, not to some a priori supposition. Attenuation is a physically realizable and measureable phenomenon and the theory and equations describing it should indicate this. There is an extremely large number of papers in the literature based on the hypothesis of complex velocities. This paper was not written to question the validity of these previous works, but rather to introduce an alternative that might be of use in specific situations.

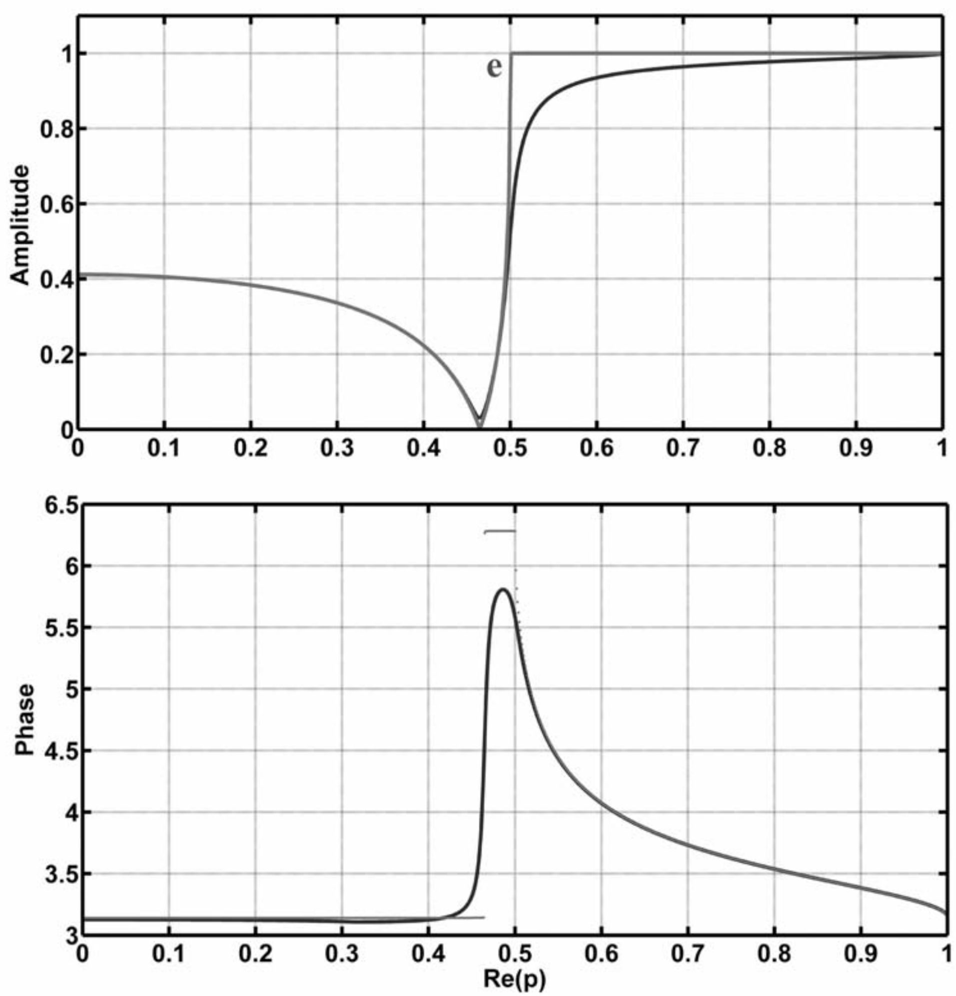

Fig. 9. The R_{11}^{SH} reflection coefficient at an interface between two medium. Both the upper (1) medium and the lower (2) media are anelastic with $Q_1 = 20$ and $Q_2 = 30$. Medium parameters are given in Table 2.

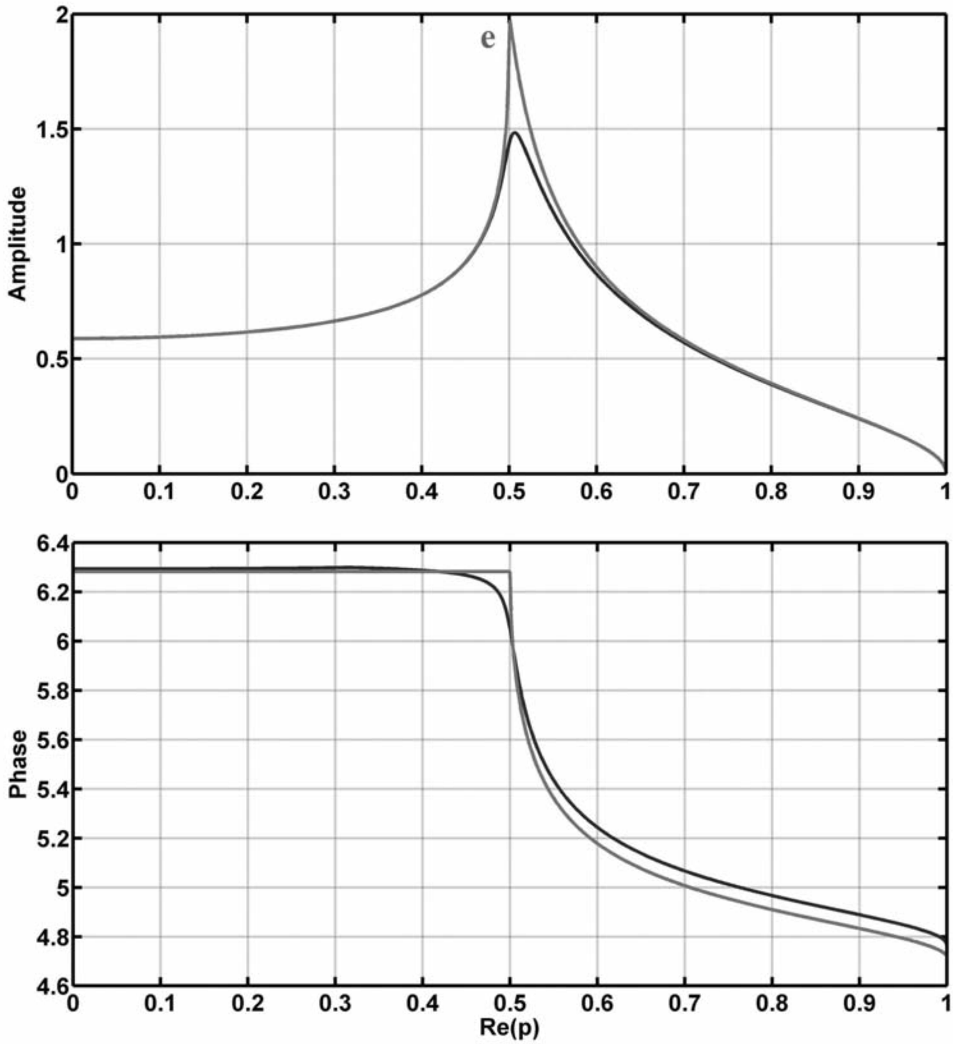

Fig. 10. The R^{SH}_{12} transmission coefficient at an interface between two anelastic media. The upper (1) medium has $Q_1 = 20$ while in the lower (2) medium $Q_2 = 30$. Medium parameters are given in Table 2.

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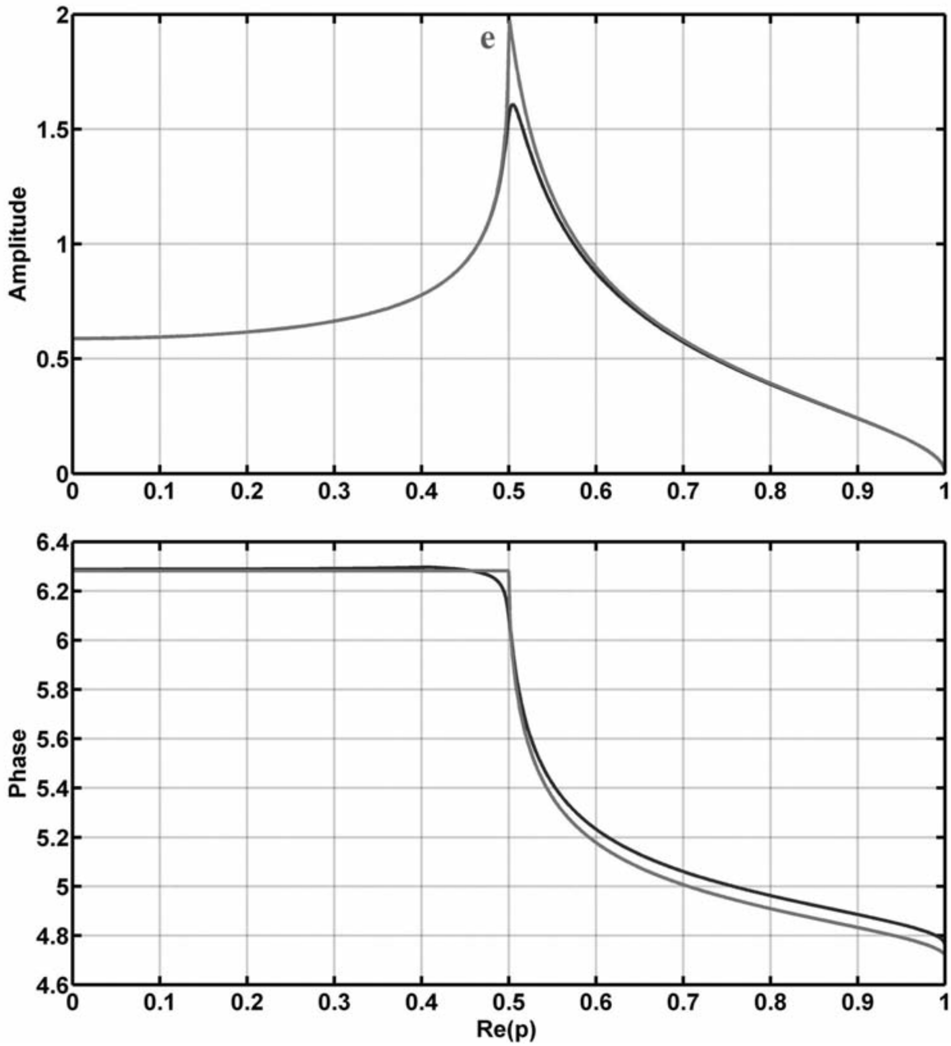

Fig. 11. The R_{12}^{SH} transmission coefficient at an interface between two anelastic media. The upper (1) medium has $Q_1 = 20$ while in the lower (2) medium $Q_2 = 50$. Medium parameters are given in Table 2.

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