RAYS IN CONSTANT-GRADIENT VELOCITY FIELDS: A TUTORIAL

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ABSTRACT

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Ray theory is a high-frequency approximation to the wave equation and can be used to calculate seismic wavefields in 3D inhomogeneous Earth models. In general, the corresponding raytracing equations have to be solved numerically. However, for certain simplified models such as, for instance, an Earth described by a constant-gradient velocity field, the solutions of the raytracing system can be obtained analytically. In this tutorial the fundamental concepts are explained, the most important equations and their solution are presented, and examples highlighting the analytical nature of describing rays in constant-gradient velocity fields are shown. This tutorial is meant to complement text books where for space reasons detailed mathematical derivations are often neglected.

KEY WORDS: ray theory, tutorial, eikonal, ray, slowness, wavefront, traveltime.

INTRODUCTION

Despite the rise of full-wavefield methods over the last decade, ray theory (see, e.g., Červený, 2001) is still frequently used in seismics. The main reason is its simplicity which often allows us to investigate and understand wave phenomena much easier than trying to look at the whole wavefield with all its complexity. Ray theory is based on a high-frequency approximation to the wave equation and leads to two fundamental alternative equations, the eikonal equation and the transport equation. While

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the eikonal equation describes the kinematics, i.e., the paths of rays and traveltimes along them, the transport equation describes the dynamics, i.e., amplitudes along the rays (see, e.g., Popov, 2002).

Here I focus primarily on the eikonal equation $(\nabla t)^2 = v^{-2}$ which is a nonlinear partial differential equation. The parameter t is the traveltime (eikonal) and v is the medium's velocity, generally varying smoothly (as required by ray theory) in all three spatial dimensions. The eikonal equation can be solved directly for t using numerical methods. This approach leads, for instance, to the well-known fast-marching eikonal solver by Sethian and Popovici (1999). Typically, however, the eikonal equation is solved in terms of characteristics (Bleistein, 1984), i.e., trajectories we call rays described by a system of ordinary differential equations, usually called the raytracing system. This system of equations can be expressed in various different ways, primarily dependent on the parameter that is used to describe a position uniquely along a ray (e.g., traveltime or arclength), see Červený (2001) for details.

For arbitrary velocity distributions in the subsurface, the raytracing system has to be solved numerically. However, in some special cases analytical solutions can be found. The simplest solution is obtained for a medium with constant velocity, $v = v_0 = \text{const.}$, where rays turn out to be straight lines. A more realistic scenario is a medium with a constant-gradient velocity field for which Slotnick (1936) published analytical solutions. Such a smooth velocity distribution in the subsurface is relatively common as velocity typically increases with depth and sedimentary structures or the Earth's mantle can be approximated fairly well in such a way. Most readers probably know that under such circumstances rays form arcs of circles (Conrad, 1922).

In the following, I will look into this scenario in more detail, derive the main analytical results in a tutorial style and present some interesting observations. Many readers will be familiar with at least some of them. Nevertheless, this compact tutorial might shed some light on well-known phenomena and provide useful insight into the behavior of rays and wavefronts in simple types of media. I will investigate a medium where velocity depends only on depth z, in particular a velocity distribution $v = v(z) = v_0 + gz$ where $g > 0$ is a constant gradient and v_0 the velocity at the source location assumed to be at the surface at $(x, z) = (0, 0)$. In such a medium rays propagate within a single vertical plane, i.e., without loss of generality we can restrict our investigations to the x-z plane and assume $y = 0$. Furthermore, with the coordinate system properly aligned as outlined above, the slowness component p_{y0} at the source is zero, i.e., $p_{x0}^2 + p_{z0}^2 = v_0^{-2}$, and p_y remains zero throughout the medium. Hence, the eikonal equation is given by $A = p_x^2 + p_z^2 = v^{-2} = u^2$ with $\mathbf{u} = \nabla t = (p_x, 0, p_z)$ being the slowness vector, and $|\mathbf{u}| = u = v^{-1}$. The slowness u, also named interval transit time, is the amount of time required for a wave to travel a certain distance; it is proportional to the reciprocal of velocity v.

RAYTRACING SYSTEM

For a medium where velocity depends only on depth, $v = v(z)$, the raytracing system can be expressed as

$$\frac{dx}{d\xi} = A^{\frac{n}{2}-1}\,p_x \; ; \quad \frac{dp_x}{d\xi} = 0 \; ; \quad \frac{dz}{d\xi} = A^{\frac{n}{2}-1}\,p_z \; ; \quad \frac{dp_z}{d\xi} = \frac{1}{n}\frac{\partial}{\partial z}\left(\frac{1}{v^n}\right) \; , \qquad (1)$$

with a fifth equation that relates the traveltime to the general variable ξ that specifies a position along the ray, namely $dt/d\xi = v^{-n}$ (Červený, 2001). The parameter $n = 0, \pm 1, \pm 2, \ldots$ actually determines the specific variable along the ray. For instance, $\xi = t$ (traveltime) for $n = 0$ and $\xi = s$ (arclength) for $n = 1$. This set of equations appears quite complex, so let us take a closer look. From the second equation of (1) it is obvious that $p_x = p_{x0} = \text{const.}$, which leads to the well-known fact that the horizontal slowness is constant in a velocity field that depends only on depth z. Typically, p_x is called p and termed the "ray parameter" or "raypath parameter" under such circumstances, as it uniquely specifies a raypath (Sheriff, 2002). The ray parameter can also be expressed as

$$p = \frac{\sin\alpha}{v} = \frac{\sin\alpha_0}{v_0} \quad , \qquad (2)$$

where α is the instantaneous angle between the ray and the vertical, and correspondingly α_0 is the take-off angle at the source. This is also known as generalized Snell's law. The eikonal equation can then be expressed as $A = p^2 + p_z^2 = v^{-2} = u^2$. Solving for p_z leads to

$$p_z = \pm\sqrt{v^{-2} - p^2} = \pm\sqrt{u^2 - p^2} \quad , \qquad (3)$$

where the plus-sign holds for downgoing waves (assuming the depth axis z points downwards) and the minus-sign holds for upgoing waves that travel in negative z-direction. The equation is valid as long as $p^2 v^2 \leq 1$, otherwise the square root becomes negative and we end up with a complex vertical slowness. Note that eq. (3) depends only on z. The parameter ξ can now be eliminated from the raytracing system and it can be expressed in terms of depth z as

$$\frac{dx}{dz} = \pm\frac{p}{\sqrt{u^2-p^2}} = \pm\frac{pv}{\sqrt{1-p^2v^2}} \;; \quad \frac{dt}{dz} = \pm\frac{u^2}{\sqrt{u^2-p^2}} = \pm\frac{1}{v\sqrt{1-p^2v^2}} \;, \tag{4}$$

assuming wave-propagation in positive x-direction. Note that looking at wave propagation in positive x-direction is no general restriction to the problem at hand as it is symmetric about the direction of the velocity gradient, i.e., the vertical axis in this case. Solutions for negative p and wave propagation in negative x-direction are in principle identical to solutions for positive p and wave propagation in positive x-direction.

DIRECT WAVE

Assuming a constant-gradient velocity field, $v = v_0 + gz$ with $g > 0$, as outlined above we can easily integrate eqs. (4) analytically to obtain solutions for $x(p,z)$ and $t(p,z)$ for the downgoing direct wave (i.e., we use the plus-sign in above equations). For distance x we obtain (see Appendix A for details)

$$x(p,z) = \frac{1}{gp}\left\{\sqrt{1-p^2v_0^2} - \sqrt{1-p^2v^2(z)}\right\} \quad , \tag{5}$$

and for the traveltime

$$t(p,z) = \frac{1}{g}\ln\frac{v(z)\left(\sqrt{1-p^2v_0^2}+1\right)}{v_0\left(\sqrt{1-p^2v^2(z)}+1\right)} \quad . \tag{6}$$

Obviously, any direct wave with $p \neq 0$ will only reach a certain finite depth $z_{\max}$. This penetration depth can be found using the generalized Snell's law from eq. (2) by setting $\alpha = 90°$ at the maximum depth $z_{\max}$ where the ray bottoms out:

$$\frac{\sin 90°}{v(z_{\max})} = \frac{1}{v_0 + gz_{\max}} = p = \text{const.} \quad \Rightarrow \quad z_{\max}(p) = \frac{1}{g}\left(p^{-1} - v_0\right) \tag{7}$$

Solving eq. (5) for p allows us to determine the ray take-off angle α_0 at the source of a downgoing wave that arrives at an arbitrary location (x,z) with $x > 0$ in the subsurface, as well as the incidence angle α at that point (see Fig. 1), provided $p^2v^2 \leq 1$:

$$p = \frac{2x}{\sqrt{g^2(x^2+z^2)^2 + 4v_0(x^2+z^2)v(z)}} = \frac{\sin\alpha_0}{v_0} = \frac{\sin\alpha}{v(z)}, \tag{8}$$

that means $\alpha_0 = \arcsin(pv_0)$ and $\alpha = \arcsin\big(pv(z)\big)$. Obviously, under the given circumstances $\alpha > \alpha_0$.

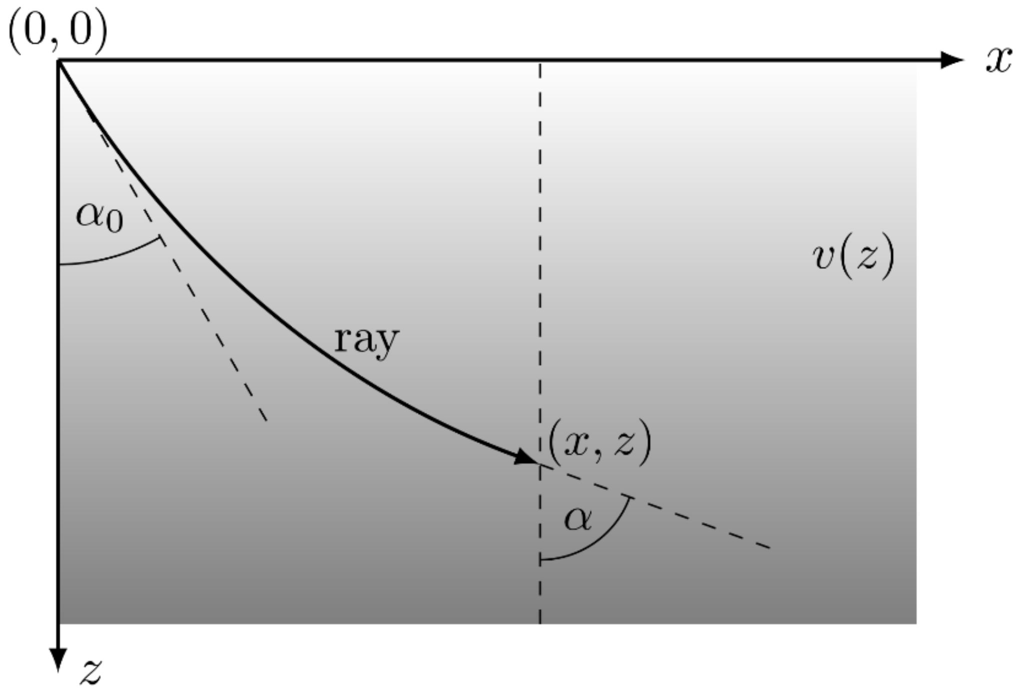

Fig. 1. The ray take-off angle α_0 at the source and the incidence angle α of the downgoing ray at an arbitrary depth point (x, z). Note that $\alpha > \alpha_0$ if $g > 0$ or, in other words, the downgoing ray bends away from the vertical.

The integrals shown in Appendix A can only be used for the downgoing wave as otherwise the negative signs in eqs. (4) have to be considered. In other words, for a receiver at the surface (assumed to be planar and horizontal) we need two integrals each to calculate the distance x and the traveltime t, one integral for the downgoing path and another integral for the upgoing path. However, as the problem is symmetric about the turning point at $z_{\max}$, we can simply use solutions (5) and (6) with $z = z_{\max}$ and multiply by 2. As $v(z_{\max}) = v_0 + gz_{\max} = p^{-1}$, we obtain

$$x(p) = \frac{2}{gp}\sqrt{1 - p^2 v_0^2} \quad , \tag{9}$$

and

$$t(p) = \frac{2}{g} \ln \frac{\sqrt{1 - p^2 v_0^2} + 1}{p v_0} \quad . \tag{10}$$

These two equations hold for a receiver at the surface, i.e., for diving waves. Using both of them we can eliminate p and obtain the traveltime as a function of offset, i.e., $t(x)$. Solving $x(p)$, eq. (9), for p and inserting the result into $t(p)$, eq. (10), leads to (see Appendix B for details)

$$t(x) = \frac{2}{g} \operatorname{arsinh}\left(\frac{xg}{2v_0}\right) \quad , \tag{11}$$

which is the traveltime-distance function for diving waves. An example is shown in Fig. 2.

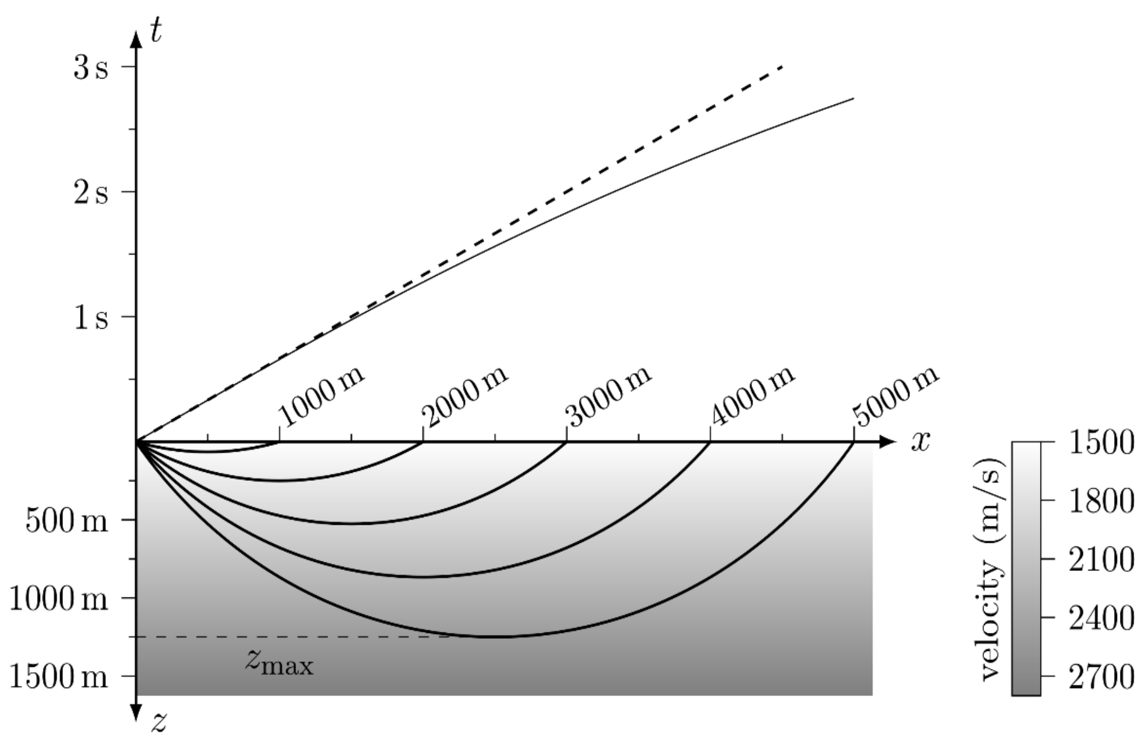

Fig. 1. Traveltime-distance curve (solid line) for diving waves given a velocity field $v(z) = 1500$ m/s $+ 0.8$ s$^{-1} z$. The traveltime of a wave traveling along the surface with $v_0 = 1500$ m/s is shown for comparison as dashed line. The lower part of the figure shows five rays associated with diving waves for offsets between 1000 and 5000 m. The maximum penetration depth of the diving wave for the longest offset of 5000 m is $z_{\text{max}} = 1250$ m.

Diving waves play an important role in applications of, for instance, full-waveform inversion. If a regional constant-gradient velocity field can be used to approximate the real Earth, eqs. (7) and (9) can be used to estimate the maximum penetration depth of diving waves for any given maximum offset x we can record during acquisition. We simply have to determine the ray parameter for the maximum offset and can subsequently determine $z_{\max}$ and the corresponding traveltime using either $t(p)$ or $t(x)$. In other words, although these equations are rather simplistic, they can be used to obtain quick estimates in real-world applications, for instance the maximum depth for which we can expect updates in applications of diving-wave full-waveform inversion.

Eqs. (5) and (6) can be used to implement, for instance, rather efficient migration methods (see, e.g., Michaels, 1977). Again, while this is no longer adequate for accurate imaging in today's challenging complex exploration environments, it is a great exercise in any applied Geophysics class and it provides a lot of insight to implement a basic Kirchhoff depth migration algorithm (Schneider, 1978) based on analytical traveltime solutions for constant-velocity or constant-gradient velocity media using eqs. (6) and (8). In this case, x would be the horizontal distance between the position of the source S or receiver R, respectively, of a seismic trace to be migrated and the image point M, yielding the one-way traveltimes t_{SM} and t_{RM}, respectively, as required by Kirchhoff migration.

CIRCLES

Although it was stated that rays in a constant-gradient velocity field form arcs of circles, it has not been shown mathematically. However, we can take eq. (5), re-arrange the terms, and express $v(z)$ explicitly in terms of z. It then reads

$$x - \frac{\sqrt{1 - p^2 v_0^2}}{gp} = -\frac{\sqrt{1 - p^2 (v_0 + gz)^2}}{gp} \quad . \tag{12}$$

Squaring the equation and again re-arranging the terms, we arrive at

$$\left[x - \frac{\sqrt{1 - p^2 v_0^2}}{gp}\right]^2 + \left[z + \frac{v_0}{g}\right]^2 = \frac{1}{(gp)^2} \quad . \tag{13}$$

This is a typical coordinate equation for a circle

$$[x - x_c]^2 + [z - z_c]^2 = r^2 \quad , \tag{14}$$

where $C = (x_c, z_c)$ is the center of the circle and r its radius. In other words, for a given ray parameter p, rays do indeed form arcs of circles with the center of the circle and its radius given by

$$(x_r, z_r) = \left(\frac{\sqrt{1 - p^2 v_0^2}}{gp}, -\frac{v_0}{g} \right) \; ; \quad r_r = \frac{1}{gp} \quad , \tag{15}$$

provided the source is located at $(x, z) = (0,0)$. It is worthwhile to note that for $g > 0$ the center of the circle is located above the surface at negative z, and it moves parallel to the surface as p changes. In fact, the center is simply located at the depth z_r where $v(z_r) = 0$, which allows for an easy geometrical interpretation. Obviously, the smaller the gradient or the smaller the take-off angle α_0 at the source, the larger the radius of the circle. And as expected, for g approaching zero (i.e., the medium turns into a homogeneous constant-velocity medium) or p approaching zero (i.e., the ray leaves the source vertically downwards), the radius approaches infinity which means rays become straight lines. Fig. 3 compares numerical and analytical results and we observe a perfect match between the numerically traced ray in the subsurface and the superimposed analytical circle.

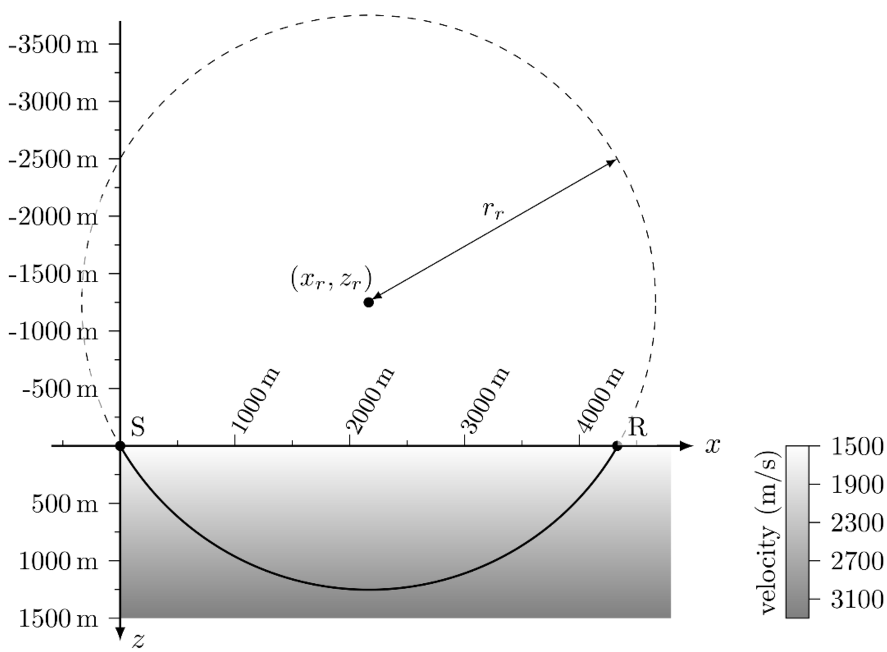

Fig. 2. Exemplary comparison of a numerically traced ray (solid bold line) with a take-off angle of 30° at the source S through the velocity field $v(z) = 1500$ m/s $+ 1.2$ s^{-1}z and an analytically calculated circle (dashed line) according to eq. (13). In this example, the center of the circle is located at $(x_r, z_r) = (2165.05 \text{ m}, -1250 \text{ m})$ and the radius is $r_r = 2500$ m. According to eq. (9) the receiver R is located at $x = 4330.1$ m (obviously twice x_r, as expected).

The length s of the ray can easily be determined from geometrical considerations. The arclength of the circular segment below the surface is given by

$$s = 2r_r \ \arcsin\left(\frac{x}{2r_r}\right) = \frac{2}{gp} \arcsin\left(\frac{gpx}{2}\right) \quad , \tag{16}$$

see, e.g., Bronshtein et al. (2015). Inserting x from eq. (9) leads to

$$s(p) = \frac{2}{gp} \arcsin\left(\sqrt{1 - p^2 v_0^2}\right) \quad . \tag{17}$$

Interestingly, the iso-line of equal traveltime, or wavefront, can for any fixed t also be described by a circle (see, e.g., Van Melle, 1948). As derived in Appendix C, this circle is given by

$$x^2 + \left[z + \frac{v_0}{g}\{1 - \cosh(gt)\}\right]^2 = \left[\frac{v_0}{g} \sinh(gt)\right]^2 \quad . \tag{18}$$

This means, the time-dependent center and radius are

$$(x_w, z_w) = \left(0, -\frac{v_0}{g}\{1 - \cosh(gt)\}\right) \ ; \quad r_w = \frac{v_0}{g} \sinh(gt) \quad . \tag{19}$$

It is worthwhile to note that x_w is always zero independent of t, i.e., the center of the wavefront circle is simply moving down (for increasing traveltimes and positive gradients) the z-axis. Fig. 4 shows an example of rays and corresponding wavefronts for certain fixed traveltimes t.

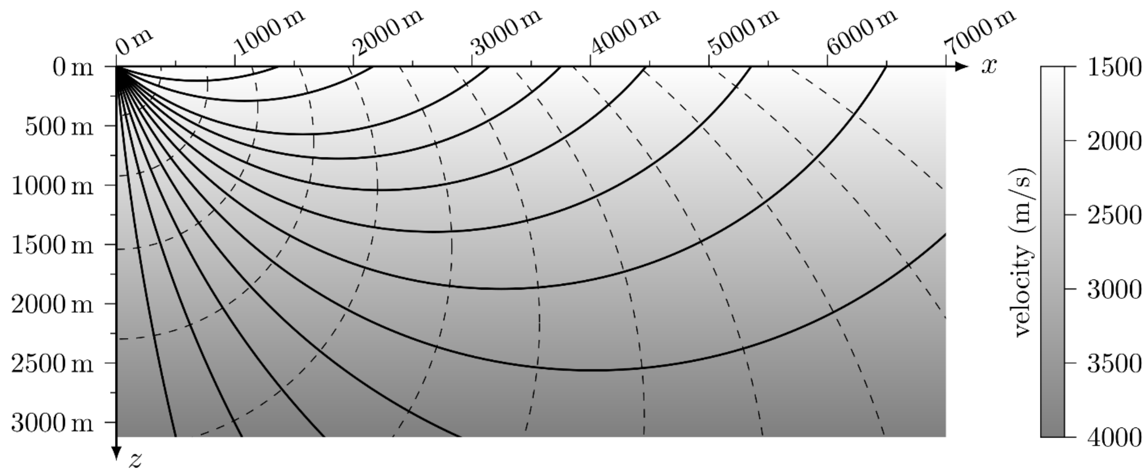

Fig. 3. Rays (solid bold lines) and wavefronts (dashed line) for a velocity field $v(z) = 1500$ m/s $+ 0.8$ s^{-1}z. Rays are plotted for take-off angles from 5° to 70° in steps of 5° until 50° and in steps of 10° beyond, using eq. (13). Wavefronts are plotted in the range $t = 0.25$ s to 3.0 s in steps of 0.25 s, using eq. (18). It can already be observed visually that wavefronts seem perpendicular to rays.

It is well known that a wavefront is perpendicular to rays in an isotropic medium for any fixed traveltime t (e.g., Červený, 2001). Having derived the circle equations for the rays and wavefronts, respectively, we can prove it by checking whether the circle defining a ray and the corresponding circle of the wavefront intersect each other orthogonally. Using the Pythagorean theorem, two circles of radii r_r and r_w whose centers are a distance d apart are orthogonal if $r_r^2 + r_w^2 = d^2$ (see, e.g., Casey, 1886). Using eqs. (15) and (19), the squared distance between the centers of the circles is given by

$$d^2 = (x_r - x_w)^2 + (z_r - z_w)^2 = \frac{1 - p^2 v_0^2}{(gp)^2} + \frac{v_0^2}{g^2}\cosh^2(gt) \quad . \quad (20)$$

Evaluating the criterion of orthogonality mentioned above,

$$r_r^2 + r_w^2 - d^2 = \frac{1}{(gp)^2} + \frac{v_0^2}{g^2}\underbrace{\sinh^2(gt)}_{=\cosh^2(gt)-1} - \left\{\frac{1 - p^2 v_0^2}{(gp)^2} + \frac{v_0^2}{g^2}\cosh^2(gt)\right\} = 0\,, \qquad (21)$$

now indeed proves that any ray and a wavefront are perpendicular under the given conditions. This holds for arbitrary, fixed traveltimes t.

REFLECTED WAVE

Let us assume there is a horizontal reflector at depth z_d. The P-wave velocity in the reflector's overburden can be described by a constant-gradient function $v(z) = v_0 + gz$ as before. The S-wave velocity in the reflector's overburden can be described by another constant-gradient function $w(z) = w_0 + hz$ with, in general, $w(z) < v(z)$. A receiver is located at point (x_R, z_R) with $z_R \geq 0$, see Fig. 5 for details.

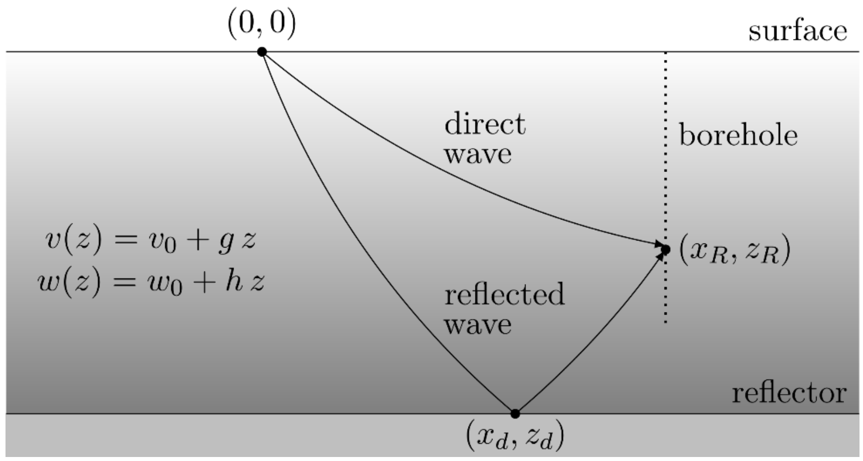

Fig. 5. Direct wave and reflected wave for a reflector at depth z_d and a receiver at (x_R, z_R), for instance in a borehole. The reflection point is given by (x_d, z_d). The P-wave velocity in the reflector's overburden is annotated by $v(z)$, the S-wave velocity by $w(z)$. Both are constant-gradient velocity functions.

We can now calculate $x(p, z_R)$ and $t(p, z_R)$ in a similar way to eqs. (5) and (6), of course provided that the ray ever hits the reflector at all (this will not be the case for all p). However, as there are two ray branches, one downgoing and one upgoing, we have to adjust the sign and use two integrals as outlined earlier. In general, x for converted waves can be calculated as follows:

$$x^{PS}(p, z_R) = \int_0^{z_d} \frac{pv}{\sqrt{1-(pv)^2}}\,dz - \int_{z_d}^{z_R} \frac{pw}{\sqrt{1-(pw)^2}}\,dz \quad . \tag{22}$$

In order to calculate x for a PP reflection, we can simply use $v(z)$ instead of $w(z)$ for the second integral, as the mode conversion happens at the reflector at depth z_d. The ray parameter p remains constant, independent of the type of reflected wave (Snell's law). This type of integral has already been solved earlier, that means we can readily write down the solutions:

$$x^{PS}(p, z_R) = \begin{aligned} &\frac{1}{gp}\left\{\sqrt{1 - p^2 v_0^2} - \sqrt{1 - p^2 v^2(z_d)}\right\} + \\ &\frac{1}{hp}\left\{\sqrt{1 - p^2 w^2(z_R)} - \sqrt{1 - p^2 w^2(z_d)}\right\} \end{aligned} \tag{23}$$

for a PS reflected wave, and for a PP reflected wave we obtain

$$x^{PP}(p, z_R) = \frac{1}{gp}\left\{\sqrt{1 - p^2 v_0^2} + \sqrt{1 - p^2 v^2(z_R)} - 2\sqrt{1 - p^2 v^2(z_d)}\right\}. \tag{24}$$

In a similar way, the traveltime for PS reflections can be calculated by

$$t^{PS}(p, z_R) = \int_0^{z_d} \frac{1}{v\sqrt{1 - (pv)^2}}\, dz - \int_{z_d}^{z_R} \frac{1}{w\sqrt{1 - (pw)^2}}\, dz \quad , \tag{25}$$

and we can again use $v(z)$ instead of $w(z)$ if we would like to obtain the solution for PP reflections. Evaluating the integrals as before, we get

$$t^{PS}(p, z_R) = \frac{1}{g} \ln \frac{v(z_d)\left(\sqrt{1 - p^2 v_0^2} + 1\right)}{v_0\left(\sqrt{1 - p^2 v^2(z_d)} + 1\right)} - \frac{1}{h} \ln \frac{w(z_R)\left(\sqrt{1 - p^2 w^2(z_d)} + 1\right)}{w(z_d)\left(\sqrt{1 - p^2 w^2(z_R)} + 1\right)} \tag{26}$$

for a PS reflected wave, and for a PP reflected wave the traveltime is given by

$$t^{PP}(p, z_R) = \frac{1}{g} \ln \frac{v^2(z_d)\left(\sqrt{1 - p^2 v_0^2} + 1\right)\left(\sqrt{1 - p^2 v^2(z_R)} + 1\right)}{v_0 v(z_R)\left(\sqrt{1 - p^2 v^2(z_d)} + 1\right)^2} \quad . \tag{27}$$

By setting $z_R = 0$, i.e., $v(z_R) \to v_0$ and $w(z_R) \to w_0$, we can simulate a receiver at the surface. The previous formulas then simplify and read

$$x^{PS}(p) = \frac{1}{gp}\left\{\sqrt{1 - p^2 v_0^2} - \sqrt{1 - p^2 v^2(z_d)}\right\} + \frac{1}{hp}\left\{\sqrt{1 - p^2 w_0^2} - \sqrt{1 - p^2 w^2(z_d)}\right\}, \tag{28}$$

$$x^{PP}(p) = \frac{2}{gp}\left\{\sqrt{1-p^2v_0^2} - \sqrt{1-p^2v^2(z_d)}\right\} \quad , \tag{29}$$

$$t^{PS}(p) = \frac{1}{g}\ln\frac{v(z_d)\left(\sqrt{1-p^2v_0^2}+1\right)}{v_0\left(\sqrt{1-p^2v^2(z_d)}+1\right)} - \frac{1}{h}\ln\frac{w_0\left(\sqrt{1-p^2w^2(z_d)}+1\right)}{w(z_d)\left(\sqrt{1-p^2w_0^2}+1\right)} \quad , \tag{30}$$

$$t^{PP}(p) = \frac{2}{g}\ln\frac{v(z_d)\left(\sqrt{1-p^2v_0^2}+1\right)}{v_0\left(\sqrt{1-p^2v^2(z_d)}+1\right)} \quad . \tag{31}$$

An interesting check of these formulas is considering a gradual move towards a constant-velocity medium, i.e., we let the gradients approach zero. Unfortunately, considering for instance the limit of eq. (29) as g approaches zero leads to an indeterminate form of the type "0/0". We can, however, use L'Hospital's rule for functions $y(g)$ and $z(g)$, which states $\lim_{g\to 0} y(g)/z(g) = \lim_{g\to 0} y'(g)/z'(g)$ if the limit of the latter actually exists (Bronshtein et al., 2015). Applying the rule to $x^{PP}(p)$ and calculating the derivatives of the numerator and denominator with respect to g shows that

$$x^{PP} = \frac{2pv_0z_d}{\sqrt{1-p^2v_0^2}} \tag{32}$$

as g approaches zero. In a similar way, we can calculate the limit of $t^{PP}(p)$ as g approaches zero which yields

$$t^{PP} = \frac{2z_d}{v_0\sqrt{1-p^2v_0^2}} \quad . \tag{33}$$

Now, solving eq. (32) for p^2 yields

$$p^2 = \frac{x^2}{v_0^2\,(x^2+4z^2)} \quad , \tag{34}$$

and inserting the latter into eq. (33) leads to

$$t^{PP}(x) = \sqrt{\frac{(2z_d)^2}{v_0^2} + \frac{x^2}{v_0^2}} = \sqrt{t_0^2 + \frac{x^2}{v_0^2}} \quad , \tag{35}$$

with $t_0 = 2z_d/v_0$ being the vertical two-way traveltime, which is the famous hyperbolic traveltime formula for a flat reflector at depth z_d with a constant-velocity overburden (see, e.g., Yilmaz, 2001).

The position x_d of the reflection point for a given offset (or receiver position at the surface) x and reflector depth z_d can be obtained by solving eq. (28) or (29), respectively, for p and then using eq. (5) and the velocity $v(z_d)$ to calculate x_d. Obviously, in case of PP reflections x_d is constant and always equals half the offset, independent of the reflector depth z_d. However, in case of PS reflections the position of x_d changes with reflector depth z_d. Unfortunately, there is no closed analytical solution when solving eq. (28) for p, i.e., we can only solve the equation numerically for specific problems.

Let us assume $v_0 = 1000$ m/s, $g = 0.6\, s^{-1}$ and $w(z) = v(z)/\sqrt{3}$, i.e., the P-wave to S-wave velocity ratio is $\sqrt{3}$. Furthermore, the source-receiver offset is fixed at $x = 1500$ m. In this case, we can numerically calculate the position of x_d as function of z_d. We can also compute the ray take-off angle at the source, the incidence angle at the reflector, or the traveltime as function of z_d. Fig. 6 shows the results.

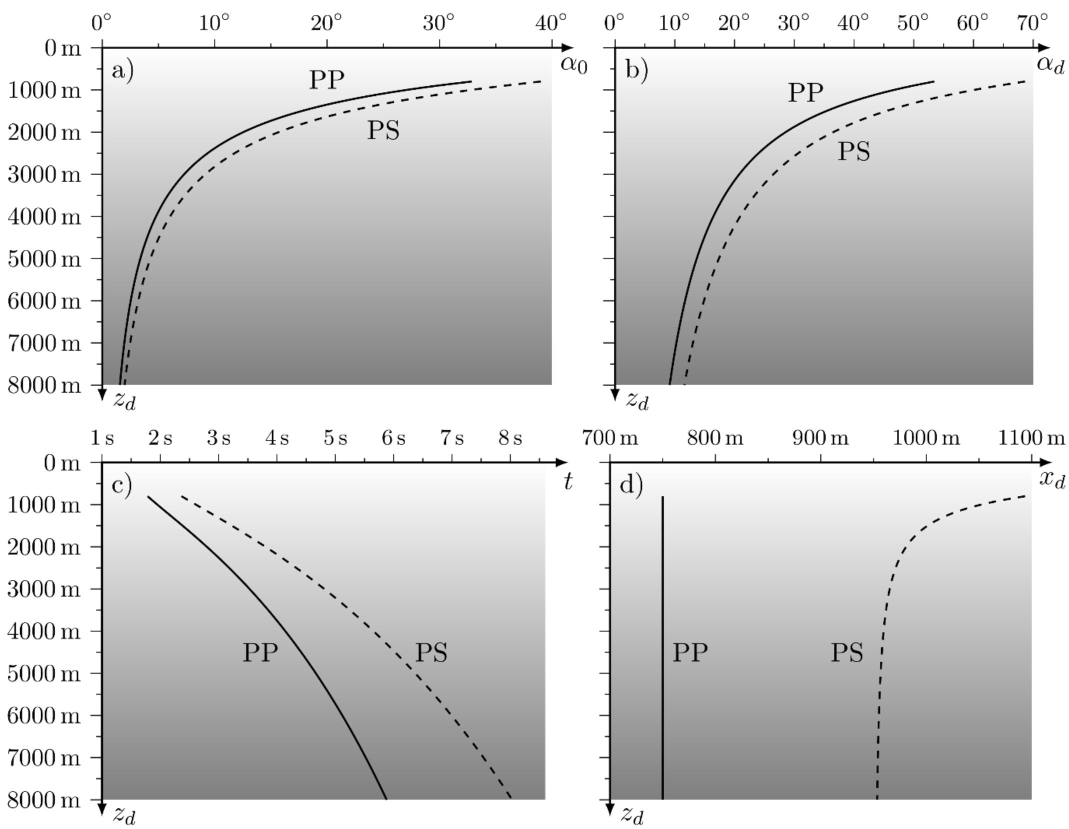

Fig. 4. Ray take-off angle at the source (a), ray incidence angle at the reflector (b), total traveltime (c) and location x_d of the reflection point (d) as function of z_d for both PP and PS reflections, assuming $v_0 = 1000$ m/s, $g = 0.6\ \mathrm{s}^{-1}$, $w(z) = v(z)/\sqrt{3}$ and offset $x = 1500$ m. The reflector depth z_d varies between 800 m and 8000 m in this example. Note that z_d is plotted on the vertical axis despite being the independent variable as it is more natural and makes these plots easier to understand.

The take-off angle at the source is always larger for PS reflected waves than for PP reflected waves. As the reflector depth approaches infinity, both take-off angles approach zero. The incidence angles at the reflector are larger than the take-off angles at the source because the rays bend away from the vertical. Again, the incidence angle for PS reflected waves is larger than for PP reflected waves. Actually, for a real reflector in the shallow we would probably be beyond the critical angle. The traveltimes for PP and PS reflected waves diverge because with increasing depth the downgoing P-part is almost identical but we spend more time traveling back up to the surface with the slower S-wave velocity. As expected, the location of the reflection point is obviously constant for the PP reflection and equals half the receiver offset, i.e., 750 m in this case. The reflection point for the PS reflection varies with receiver depth and approaches 950 m as the depth z_d increases. This change in the position of reflection points has implications on the binning process of PS-converted waves where typically no common-midpoint (CMP) binning is performed but common-conversion point (CCP) binning. The position x_d of the CCP for $z_d \rightarrow \infty$ and a fixed P-wave to S-wave velocity ratio is usually called asymptotic conversion point (ACP) in the literature (e.g., Yilmaz, 2001). Slawinski and Slawinski (1999) show how considerations presented in this section can affect amplitude-versus-angle (AVA) analyses.

NONVERTICAL CONSTANT-GRADIENT VELOCITY FIELDS

The calculations presented so far only hold for vertical velocity gradients and a source at position $(x, z) = (0, 0)$. However, the results can easily be generalized to nonvertical constant-gradient velocity fields and alternative source positions. Shifting the source position is, mathematically, simply introducing a translation of the coordinate system. Nonvertical constant gradients require a rotation of the coordinate system such that the direction of the velocity gradient equals the rotated depth axis z'. Furthermore, we have to make sure that the initial slowness vector at the source has no y'-component in the rotated system. Honoring these conditions should provide a rotated coordinate system in which rays propagate only in the x'-z' plane and standard formulas as derived here can be used. Upon successful calculations the results then have to be transformed back to the unrotated coordinate system. Details and the required rotation matrices can be found in publications by, for instance, Kim and Lee (1997) or Baig et al. (2001).

RADIALLY SYMMETRIC EARTH

When the problems discussed here are scaled up and we consider the whole Earth or the Earth's mantle, we have to take into account that the

Earth is a sphere. Actually, on a global scale the Earth's velocity field can be described by a radially symmetric function fairly well, i.e., $v = v(r)$ where r is the radius. Obviously, under such circumstances it makes more sense to use a spherical rather than a Cartesian coordinate system. The concepts presented in this paper remain the same, the mathematics changes though. Instead of the horizontal distance x, we now deal with the angular distance Δ which, for a source and a receiver at the surface of the sphere, is given by

$$\Delta(p) = 2 \int_{r_{\min}}^{r_e} \frac{pv(r)}{\sqrt{r^2 - p^2 v^2(r)}} \frac{dr}{r} \quad , \tag{36}$$

and the corresponding traveltime by

$$t(p) = 2 \int_{r_{\min}}^{r_e} \frac{r^2}{v(r)\sqrt{r^2 - p^2 v^2(r)}} \frac{dr}{r} \quad , \tag{37}$$

with $p = (r \sin \alpha)/v(r)$ being the ray parameter, r_e the Earth's radius, and $r_{\min}$ the radius at the point the ray bottoms out (see Fig. 7 and, e.g., Aki and Richards, 2002). Note that the ray parameter here has a different unit compared to the ray parameter for a flat Earth, although typically the same symbol is used in the literature and the spherical- and flat-Earth ray parameters are not distinguished explicitly.

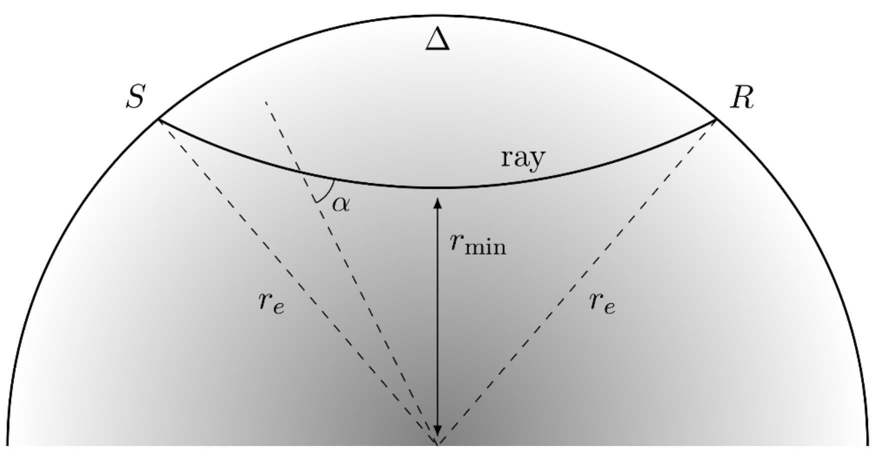

Fig. 5. A ray in a radially symmetric spherical Earth where velocity increases with depth. The angular distance between the source S and receiver R is called Δ. The parameter r_e is the Earth's radius, $r_{\min}$ is the minimal radius at the point the ray bottoms out. It corresponds to $z_{\max}$ for a flat Earth. The angle α is the angle between the radial direction and the local direction of the ray. Obviously, $\alpha = 90°$ when $r = r_{\min}$.

The equations for a spherical Earth can be obtained when replacing $x(p)$ by $r_e\,\Delta(p)$, z by $r_e\ln(r_e/r)$, $v(z)$ by $r_e\,v(r)/r$, and p by p/r_e. The solutions for spherical Earth problems can therefore be obtained immediately from the solutions for a flat Earth by above mentioned change of variables (see, e.g., Aki and Richards, 2002, for details). This is also known as the "Earth-flattening transformation" in the literature (e.g., Shearer, 2009). The situation in general is well known among the seismological community and typically studied in detail in Seismology courses. The above statement also holds in the special case presented in this paper where the velocity field is described by a constant-gradient function. In the case of a radially symmetric sphere, the gradient would then be defined with respect to the radius r rather than depth z.

SUMMARY AND CONCLUSIONS

Although rays and simple types of media such as constant-gradient Earth models seem outdated when looking at today's complex imaging problems and full-wavefield methods, they still provide great insight into fundamental concepts and facilitate our understanding of wave phenomena. In this light I presented a tutorial on rays in constant-gradient velocity fields, derived the fundamental equations and their mathematical solutions and pointed out basic applications and relationships such as the famous normal-moveout equation that remain important even in today's times of advanced processing and imaging. Furthermore, I presented a few simple graphical examples that each reader will be able to reproduce. This tutorial therefore complements text books and earlier publications by providing a compact but comprehensive overview of the subject at hand.

ACKNOWLEDGMENTS

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APPENDIX A

DISTANCE AND TRAVELTIME FUNCTIONS OF DIRECT WAVES

Distance and traveltime of the direct wave in a constant-gradient velocity field can be obtained analytically by integrating eqs. (4):

$$\begin{aligned} x(p,z) &= \int_0^z \frac{pv}{\sqrt{1-(pv)^2}}\,dz' = \int_0^z \frac{p\,(v_0+gz')}{\sqrt{1-p^2(v_0+gz')^2}}\,dz' \\ &= -\frac{1}{gp}\sqrt{1-p^2(v_0+gz')^2}\,\Big|_0^z = \frac{1}{gp}\left\{\sqrt{1-p^2v_0^2}-\sqrt{1-p^2v^2(z)}\right\} \end{aligned} \qquad (A-1)$$

and

$$\begin{aligned} t(p,z) &= \int_0^z \frac{1}{(v_0+gz')\sqrt{1-p^2(v_0+gz')^2}}\,dz' = \frac{1}{g}\ln\frac{v_0+gz'}{\sqrt{1-p^2(v_0+gz')^2}+1}\,\Big|_0^z \\ &= \frac{1}{g}\ln\frac{v(z)\left(\sqrt{1-p^2v_0^2}+1\right)}{v_0\left(\sqrt{1-p^2v^2(z)}+1\right)} \quad , \end{aligned} \qquad (A-2)$$

using, for instance, integral tables of Bronshtein et al. (2015) to solve the integrals.

APPENDIX B

TRAVELTIME FUNCTION OF DIVING WAVES

Given are the two equations for direct waves

$x(p) = \frac{2}{pg}\sqrt{1-p^2v_0^2}$ and

$$t(p) = \frac{2}{g}\ln\frac{\sqrt{1-p^2v_0^2}+1}{pv_0} = \frac{2}{g}\ln\left(\sqrt{\frac{1}{p^2v_0^2}-1}+\frac{1}{pv_0}\right),$$

see eqs. (9) and (10) above. Solving the equation $x(p)$ for p leads to

$$p = \pm\frac{2}{\sqrt{x^2g^2+4v_0^2}} \quad . \qquad (B-1)$$

As we are looking for wave propagation in positive x-direction, i.e., $0 \le \alpha_0 \le \pi/2$, we can ignore the negative sign and choose $p = 2/\sqrt{x^2 g^2 + 4v_0^2}$. Eliminating p from $t(p)$ leads to

$$\begin{aligned} t(x) &= \frac{2}{g} \ln\left(\sqrt{\frac{x^2 g^2 + 4v_0^2}{4v_0^2} - 1} + \frac{\sqrt{x^2 g^2 + 4v_0^2}}{2v_0} \right) \\ &= \frac{2}{g} \ln\left(\frac{xg}{2v_0} + \sqrt{\left(\frac{xg}{2v_0}\right)^2 + 1} \right) = \frac{2}{g} \ln\left(\xi + \sqrt{\xi^2 + 1}\right) \quad \text{with} \quad \xi = \frac{xg}{2v_0} \quad . \end{aligned} \tag{B-2}$$

As $\ln\left(\xi + \sqrt{\xi^2 + 1}\right) = \operatorname{arsinh}(\xi)$ (see, e.g., Bronshtein et al., 2015), we finally obtain

$$t(x) = \frac{2}{g} \operatorname{arsinh}\left(\frac{xg}{2v_0}\right) \quad , \tag{B − 3}$$

which is the traveltime-distance function (11) shown above.

APPENDIX C

WAVEFRONT CIRCLES

Obtaining the circle equation for wavefronts in constant-gradient velocity fields involves quite a lot of algebraic manipulations, i.e., the details are typically skipped in textbooks. Here, one way of deriving eq. (18) is presented that is based on eliminating the variable p from eqs. (5) and (6). Van Melle (1948) derives the circle equation for wavefronts, for instance, purely based on geometrical considerations.

Squaring eq. (5) and multiplying with g^2 leads to

$$\begin{aligned} g^2 x^2 &= \frac{1}{p^2}\left(1 - p^2 v_0^2 + 1 - p^2 v_z^2 - 2\sqrt{1 - p^2 v_0^2}\sqrt{1 - p^2 v_z^2}\right) \\ &= -v_0^2 - v_z^2 + \frac{2}{p^2}\left(1 - \sqrt{1 - p^2 v_0^2}\sqrt{1 - p^2 v_z^2}\right) , \end{aligned} \tag{C − 1}$$

where, for simplicity, the abbreviation $v_z = v(z)$ is used. By reorganizing the terms, we obtain

$$\frac{g^2x^2 + v_0^2 + v_z^2}{2v_0v_z} = \frac{1}{p^2v_0v_z}\left(1 - \sqrt{1 - p^2v_0^2}\sqrt{1 - p^2v_z^2}\right) \quad . \qquad (C-2)$$

Now looking at eq. (6) again, we can also write it as

$$\begin{aligned} gt \quad &= \ln\frac{v_z\left(\sqrt{1 - p^2v_0^2} + 1\right)}{v_0\left(\sqrt{1 - p^2v_z^2} + 1\right)} = \ln\frac{\sqrt{1 - p^2v_0^2} + 1}{pv_0} - \ln\frac{\sqrt{1 - p^2v_z^2} + 1}{pv_z} \\ &= \cosh^{-1}\left(\frac{1}{pv_0}\right) - \cosh^{-1}\left(\frac{1}{pv_z}\right) \quad , \qquad (C-3) \end{aligned}$$

where $\cosh^{-1}$ is the area hyperbolic cosine (or inverse hyperbolic cosine). The latter step is not obvious. However, $\ln\frac{\sqrt{1-p^2v^2}+1}{pv}$ can be written as $\ln\left(\sqrt{\left(\frac{1}{pv} - 1\right)\left(\frac{1}{pv} + 1\right)} + \frac{1}{pv}\right)$ which according to the theorem $\ln\left(\sqrt{(\xi - 1)(\xi + 1)} + \xi\right) = \cosh^{-1}(\xi)$ (see, e.g., Bronshtein et al., 2015) equals $\cosh^{-1}\left(\frac{1}{pv}\right)$.

Using the theorem

$$\cosh^{-1}(\xi_1) - \cosh^{-1}(\xi_2) = \cosh^{-1}\left(\xi_1\,\xi_2 - \sqrt{(\xi_1^2 - 1)(\xi_2^2 - 1)}\right)$$

(see again, e.g., Bronshtein et al., 2015) eq. (C-3) can be further simplified to read

$$gt = \cosh^{-1}\left(\frac{1}{p^2v_0v_z}\left[1 - \sqrt{1 - p^2v_0^2}\sqrt{1 - p^2v_z^2}\right]\right) \quad . \qquad (C-4)$$

Using eq. (C.2), the argument of the area hyperbolic cosine can now be replaced and thereby p is eliminated:

$$gt = \cosh^{-1}\left(\frac{g^2x^2 + v_0^2 + v_z^2}{2v_0v_z}\right) \quad . \qquad (C-5)$$

Taking the hyperbolic cosine of this equation and substituting $v_z = v_0 + gz$ leads to

$$\cosh(gt) = \frac{g^2x^2 + v_0^2 + (v_0 + gz)^2}{2v_0(v_0 + gz)} \quad . \qquad (C-6)$$

Multiplying this equation by $2v_0(v_0 + gz)$, adding $v_0^2\big(1 - \cosh(gt)\big)^2/g^2$ to both sides of the equation, and rearranging the terms leads to

$$x^2 + z^2 + \frac{2v_0 z}{g}\big(1 - \cosh(gt)\big) \quad + \frac{v_0^2}{g^2}\big(1 - \cosh(gt)\big)^2 =$$
$$\frac{2v_0^2}{g^2}(\cosh(gt) - 1) + \frac{v_0^2}{g^2}\big(1 - \cosh(gt)\big)^2.$$

$(C-7)$

Simplifying the terms and using the identity $\cosh^2(gt) - 1 = \sinh^2(gt)$ finally leads to

$$x^2 + \left[z + \frac{v_0}{g}\{1 - \cosh(gt)\}\right]^2 = \left[\frac{v_0}{g}\sinh(gt)\right]^2 \quad , \qquad (C-8)$$

which is eq. (18) for the wavefront circle shown above. The operations performed here are valid under the original assumptions being made (e.g., $g > 0$, $x > 0$, and $v_0 > 0$).